VERIFICATION OF EILENBERGE- MACLANE SPACES USING OBSTRUCTION THEORY AND INDUCTION ARGUMENT

Okwonu, F.Z. and Atonuje, A.O.
Department of Mathematics and Computer Science, Delta State University, Abraka

ABSTRACT

By using the obstruction theory and induction argument, we show that $\prod_q(X_{n+k}) \to \prod_q(X_{n+k+1})$ is an isomorphism for $q < n+k,$ and $\prod_{n+k}(X_{n+k}) \to \prod_{n+k}(X_{n+k-1})$ is surjective. With kernel K (G,n), the subgroup generated by $\partial u : u \in C$ is the subgroup generating $\prod = \prod_{n+k}(X_{n+k})$ and we establish that the homomorphism is zero. Since it is surjective, implies that $\prod_{n+k}(X_{n+k+1}) = 0.$ Hence X_{n+k+1} has the required properties on its homotopy groups making an Eilenberg – Maclane space.

KEY WORDS: Fibration, Obstruction, Surjective, Fiber Bundle.

1.0 INTRODUCTION.

Homotopy groups are used in algebraic topology to classify topological spaces. In particular, two mappings are said to be homotopic if one can be continuously buckled into the other. Homotopy groups are applicable in the study of topological spaces. It is a well known fact that if two structures in a topological space do not have the same homotopy group, then their topological structure is different.

Eilenberg-Maclane spaces are building blocks for homotopy theory. The spaces are important in algebra for construction and calculation of homotopy groups of spheres and cohomology operations. Also, every topological space has the homotopy type of an iterated fibration of Eilenberg – Maclane spaces to the homotopy groups of spheres.

In the present paper, we verify the Eilenberg- Maclane spaces using obstruction theory as well as induction argument. This algebra is robust for application in topological spaces. Let $V_B S^n$ be a wedge of spheres whose nth - homotopy group $\Re_B$ is isomorphic to its homology and is a free Abelian consequence of Hurewicz theorem (Hurewicz 1955). By theorem 4.1 we have that

$$\prod_q(X_{n+k}) \to \prod_q(X_{n+k+1}),$$

is an isomorphism for $q < n+k,$ and

$$\prod_{n+k}(X_{n+k}) \to \prod_{n+k}(X_{n+k-1}),$$

is surjective, with kernel the subgroup generated by $\partial u ; u \in C$. Since this subgroup generates equation (4.2), we see that the homomorphism is zero since it is surjective, implies that

$$\prod_{n+k}(X_{n+k+1}) = 0.$$

Hence X_{n+k+1} has the required property on its homotopy groups.

The rest of this paper is organized as follows. Fibration and its application to homotopy stability of the orthogonal and unity group are presented in sections 2 and 3. Obstruction theory and the existence of Eilenberg-Maclane spaces are considered in section 4. Conclusions are drawn in section 5.

2. 0 FIBRATION:

Let ϑ be CW-complex. Since its cellular k -chain $C_k(\vartheta)$ is the free abelian group generated by the k - dimensional cells in $\vartheta,$ then the cochains with coefficients in group G_{grp} are define by

$$C_k(\vartheta, G_{grp}) = Hom(C_k(\vartheta), G_{grp}). \quad (2.1)$$

Definition 2.1

A serre fibration is one that is surjective and is a continuous map $p_z : E_p \to B_l$ that satisfies the homotopy lifting property for CW-complex (Milnor 1959; Cohen 1998). That is, if ϑ is any CW-complex and $H : \vartheta \times 1 \to B_l$ is a continuous homotopy such that $H_0 : \vartheta \times \{0\} \to B_l$ factor through a map $H_0 : \vartheta \to E_p,$ such that there exist a lifting $\hat{H} : \vartheta \times 1 \to E_p$ that extends H_0 on $\vartheta \times \{0\},$ as such the diagram below commute

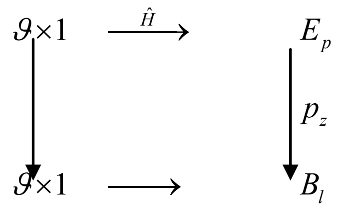

Fig. 1

Proposition 2.1 (Cohen, 1998)

Let ϑ be any connected space with basepoint $\mu_0 \in \vartheta.$ Let $p\vartheta$ be the space of base paths in $\vartheta.$ That is $p\vartheta = \{\alpha_p : I \to \vartheta : \alpha_p(0) = \mu_0\}.$ The path space $p\vartheta$ is topologozed using the compact – open function space topology define by

$$\rho : p\vartheta \to \vartheta. \quad (2.2)$$

Then the path space is a contractable space and the map equation (2.2) is a fibration, whose fiber at $\mu_0, p^{-1}(\mu_0)$ is the loop space $\varpi\vartheta.$

Proof:

The fact that $p\vartheta$ is contractable is straight forward. For a null homotopy of the identity map, one can take the map

$$H_p : p\vartheta \times 1 \to p\vartheta. \quad (2.3)$$

defined by

$$H_p(\mu_0, s)(k) = \mu_0((1-s)k.$$

To show that equation (2.2) is a fibration, we must show that it satisfies the homotopy lifting propery (HIP).

Let

$$H : Y \times 1 \to \vartheta,$$
$$H_0 : \vartheta \to p\vartheta,$$

be maps in which every operation commute. Then a homotopy lifting is define as follows

$$\hat{H} : Y \times 1 \to p\vartheta.$$

Such that $(y_p, s) \in Y \times 1,$ this implies that the path can be stated as follows

$$\hat{H}(y_{p,}s) : 1 \to \vartheta,$$

$$\hat{H}(y_{p,}s)(k) = \begin{cases} H_0(y_p)(2k/2-s), k \in [0, 2-s/2] \\ \\ H(y_p, 2k-2+s, k \in [2-s/2, 1] \end{cases}.$$

The above equation defines the continuous map and satisfies the boundary condition (Atiyah 1966; Cohen 1998).

$$\hat{H}(y_p, 0)(k) = H_0(y_p, k),$$
$$\hat{H}(y_p, s)(0) = \mu_0,$$
$$\hat{H}(y_p, s)(1) = H(y_p, s).$$

Theorem 2.1 (Atiyah (1966))

Every continuous map $f : X \to Y$ is homotopic to a fibration in the sense that there exist a fibration $\bar{f} : \bar{X} \to Y,$ (2.4)

and a homotopy equivalence

$$h : X \xrightarrow{\simeq} \bar{X}, \quad (2.5)$$

making the diagram commute,

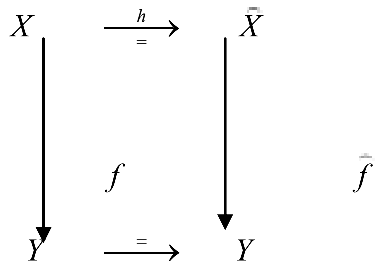

Fig. 2

Proof

Define $\bar{X}$ to be the space

$$\bar{X} = \{(x,\alpha) \in X \times Y' : \alpha(0) = x\},$$

where Y' denotes the space of continuous maps (Cohen 1998)

$$\alpha = [0,1] \to Y,$$

gives the compact open topology. Equation (2.4) is defined by

$$f(x,\alpha) = \alpha(1).$$

The fact that equation (2.4) is a fibration proved in proposition 2.1, define equation (2.5) by

$$h(x) = (x, e_x) \in \bar{X},$$

where $e_x(t) = x,$ is the constant path at $x \in X.$ Fig.3 commutes, that is

$$\bar{f} \times h = f.$$

Suppose that from Fig. 3 and by transforming the parameters we have

$$g : \bar{X} \to X,$$

defined by

$$g(x,\alpha) = x,$$

where $g \times h$ is the identity map on $X.$ We observe that $h \times g$ is homotopic to the identity on $\bar{X},$ by considering the homotopic

$$f : \bar{X} \times 1 \to \bar{X},$$

where

$$f((x,\alpha)s) = (x, \alpha_s),$$
$$\alpha_s : 1 \to X. \quad (2.6)$$

equation (2.6) is the path $\alpha_s(t) = \alpha(st),$ this implies that $\alpha_0 = e_x,$ and $\alpha_1 = \alpha.$ Thus, f is a homotopy between $h \times g$ and the identity map on $\bar{X}.$ Thus, h is a homotopy equivalence.

The following results can be found in Hurewicz (1955)

Theorem 2.2

Let $B \subset X$ be a subspace containing the basepoint $x_0 \in B.$ Then we have a long exact sequence in homotopy groups

$$\ldots \xrightarrow{\rho_*} \Pi_n(A) \xrightarrow{i_*} \Pi_n(\alpha) \xrightarrow{j_*} \Pi_n(X,B) \xrightarrow{\rho_*} \Pi_{n-1}(B) \ldots \to \Pi_1(B) \xrightarrow{i_*} \Pi_1(X).$$

Proof: omitted.

Theorem 2.3

Let $p : D \to A$ be a fibration over a connected space A with fiber $F.$ We assume that the basepoint of $D,$ is contained in $F,$ $e_0 \in F,$ and $p(e_0) = a_0$ is the basepoint in $A.$ Let $i : F \to D$ be the inclusion of the fiber. Then there is a long exact sequence of homotopy groups (Hurewicz 1955)

$$\ldots \xrightarrow{\psi_*} \Pi_n(F) \xrightarrow{i_*} \Pi_n(D) \xrightarrow{p_*} \Pi_n(A) \xrightarrow{\psi_*} \Pi_{n-1}(F$$

Proof: omitted.

3.0 APPLICATION TO HOMOTOPY STABILITY OF THE ORTHOGONAL AND UNITY GROUP

Corollary 3.1

The sphere bundles

$$V^{2k-1} \to Cu(k-1) \to Cu(k),$$
$$V^{k-1} \to Co(k-1) \to Co(k),$$

are isomorphic to the unit sphere bundles of the universal vector bundle σ_k over $Cu(k)$ and $Co(k)$ respectively.

Proof:

We apply Grassmannian models for $Cu(k)$ and $Co(k),$ then their relation to the sphere bundles (Lewis 1986) can be seen explicitly in the following ways. Consider embedding

$$i: G_{rk-1}(\Re^N) \to G_{rk}(\Re^N \times \Re) = G_{rk}(\Re^{N+1}),$$

defined by

$$(W \subset \Re^N) \to (W \times \Re \subset \Re^N \times \Re),$$

as $N \to \infty,$ this map becomes a model for the inclusion $Co(k-1) \to Co(k),$

for $W \in G_{rk-1}(\Re^N).$

Consider the vector $(0,1) \in W \times \Re \subset \Re^N \times \Re,$ which is a unit vector, and so is an element of the fiber of the unit sphere bundle $S(\sigma_k)$ over $W \times \Re.$ This association defines a map,

$$j: G_{rk-2+1}(\Re^N) \to S(\sigma_k),$$

which lift

$$i: G_{rk-2+1}(\Re^N) \to G_{rk-1+1}(\Re^{N+1}).$$

By taking a limit over $N,$ we get a map

$$j: Co(k-1) \to S(\sigma_k).$$

To define homotopy inverse

$$\varsigma: S(\sigma_k) \to Co(k-1), \qquad (3.1)$$

we apply finite Grassmannian level. Let $(Q,q) \in S(\sigma_k),$ then the unit sphere bundle over $G_{rk}(\Re^m),$ thus, $Q \subset \Re^m$ is m -dimensional subspace and $q \in Q$ is a unit vector. Let $Qq \subset Q$ denote the orthogonal complement to the vector $q \in Q.$ Thus, $Qq \subset Q \subset \Re^{m-1}$ is an $m-1$ -dimensional subspace. This association defines a map

$$\varsigma: S(\sigma_m) \to G_{rm-1}(\Re^m), \qquad (3.2)$$

and by taking the limit over $m,$ equation (3.2) defines a map which implies that equations (3.1) and (3.2) are inverse of each other.

Theorem 3.1 (Hilton and Wylie, 1967))
The inclusion maps

$$i: o(k) \to o(k+1),$$
$$u(k) \to u(k+1),$$

Which induce isomorphism is a homotopy group through dimensions $k-2$ and $2k-1$ respectively. Also induces maps on classifying spaces

$$\hat{C}_i: \hat{C}o(k) \to \hat{C}o(k+1),$$
$$\hat{C}u(k) \to \hat{C}u(k+1),$$

induces isomorphism in homotopy groups through dimensions $k-1$ and $2k$ respectively.

Proof:

The first two statements follows from the existence of fiber bundle corollary 3.1

$$o(k) \to o(k+1) \to S^k,$$
$$u(k) \to u(k+1) \to S^{2k+1}.$$

The connectivity of sphere and by applying the exact sequence in homotopy groups to these fiber bundles (Lewis 1986; Atiyah 1966). The second statement follows from the considerations from corollary 3.1 the fiber bundles

$$S^k \to \hat{C}o(k) \to \hat{C}o(k+1),$$
$$S^{2k+1} \to \hat{C}u(k) \to \hat{C}u(k+1).$$

4.0 OBSTRUCTION THEORY AND THE EXISTENCE OF EILENBERG – MACLANE SPACES:

The following results form the extension of obstruction theory to establish the properties of Eilenberg – Maclane spaces:

Theorem 4.1

Let X be a simply connected CW-complex and let $f: S^k \to X,$ be a map. Let X' be the mapping cone of $f.$ That is $X' = X \cup fD^{n+1},$ which denotes the union of X with a disk D^{n+1} glued along the boundary sphere

$S^k = \sigma D^{n+1}$,

via f, such that $t \in S^k$ with $f(t) \in X$. Let $i : X \to X'$,

be the inclusion. Then

$i_* : \prod_k(X) \to \prod_k(X')$, (4.1)

is surjective with kernel equal to the cycle subgroup generated by

$$[F] \in \prod_k(X).$$

Proof:

Let $g : S^q \to X'$,

Consider the element in $\prod_q(X')$ with $q \le k$. By cellular approximation theorem (Milnor 1959), g is homotopic to a cellular map and one whose image lies in the q -skeleton of X'. But for $q \le k$, then the q -skeleton of X' is the q -skeleton of X. This implies that equation(4.1) is surjective for $q \le k$. Assume $q \le k-1$, then

$g : S^q \to X \subset X'$,

is null homotopic. Any null homotopy that is extension to the disk

$$G : D^{q+1} \to X',$$

can be assumed to be cellular and hence has image in X. Thus, $q \le k-1$, equation (4.1) is an isomorphism. By the exact sequence in homotopy groups of the pair (X', X), implies that the pair (X', X) is k -connected. By applying Hurewicz theorem (Hurewicz 1955) we have

$\prod_{k+1}(X', X) \cong H_{k+1}(X', X) = H_{k+1}(X \cup fD^{k+1}, X)$,

by analyzing the cellular chain complex for computing $H_*(X')$ for is possible if

$f : S^k \to X$,

is zero in homotopy and zero otherwise. The generator $\partial \in \prod_{k+1}(X', X)$,

is represented by the map of pairs given by the inclusion

$$\partial : (D^{k+1}, S^k) \to (X \cup fD^{k+1}, X),$$

and in the long exact sequence in homotopy groups of the pair (X', X),

$$\cdots \to \prod_{k+1}(X', X) \xrightarrow{\partial_*} \prod_k(X) \xrightarrow{i_*} \prod_k(X') \to \cdots,$$

we have

$\partial_*(\partial) = [f] \in \prod_k(X)$.

Thus, equation (4.1) is surjective with kernel generated by $[f]$.

Theorem 4.2

Let G be any abelian group and n an integer with $n \ge 2$. Then there exist a space $K(G, n)$ with

$$\prod_k(K(G,n)) = \begin{cases} G, & if \quad k = n \\ 0, & otherwise. \end{cases}$$

Proof:

let G be fix group and $n \ge 2$. Let $\{\partial_\alpha : \alpha \in A\}$ be a set of generator of G, where A denotes the indexing set for these generator. Let $\{\theta_\beta : \beta \in B\}$, be corresponding set of relations. The group G is isomorphic to the free abelian group F_A generated by A, modulo the subgroup $\Re_B$ generated by $\{\theta_\beta : \beta \in B\}$.

Consider the wedge of spheres $V_A S^n$ indexed on the set A. By the Hurewicz theorem (Hurewicz 1955)

$$\prod_n(\bigvee_A S^n) \cong H_n(\bigvee_A S^n) \cong F_A.$$

Since the group $\Re_B$ is a subgroup of a free abelian group and hence is it free abelian group. Let $V_B S^n$ be a wedge of spheres whose nth - homotopy group is $\Re_B$ is isomorphic to its homology and is a free abelian consequence of Hurewicz theorem. There exists a natural map

$$j: \bigvee_B S^n \to \bigvee_A S^n,$$

which on the level of the homotopy group Π_n is the inclusion $\Re_B \subset F_A.$ Let X_{n+1} be the mapping cone of $j.$ Then

$$X_{n+1} = \bigvee_A S^n \bigcup_j \bigcup_B D^{n+1},$$

where the disk D^{n+1} corresponding to the generator in $\Re_B$ is attached through the map

$$S^n \to V_A S^n,$$

giving the corresponding element in

$$\Pi_n(V_A S^n) = F_A.$$

Then by applying theorem 4.1 one cell at a time, we observed that X_{n+1} is an $n-1$ -connected space and $\Pi_n(X_n)$ is generated by F_A modulo the subgroup $\Re_B$ which implies that

$$\Pi_n(X_{n+1}) \cong G.$$

Assume inductively, we can construct the space X_{n+k} with

$$\Pi_q(X_{n+k}) = \begin{cases} 0 & if \quad q < n, \\ G & if \quad q = n, \\ 0 & if \quad n < q \le n+k-1. \end{cases}$$

We start the inductive argument with $k = 1,$ by the construction of the space X_{n+1} above. We assume by constructing $X_{n+k},$ we need to show how to construct X_{n+k-1} with these properties. By induction we let $k \to \infty$ and X_∞ will be a model for $K(G,n).$ Let

$$\Pi = \Pi_{n+k}(X_{n+k}), \qquad (4.2)$$

with generating set $\{\partial_u : u \in c\},$ where c is the indexing set. Let F_c be the free abelian group generated by the element in this generating set. Let $V_u \in S_u^{n+k},$ be a wedge of sphere indexed by the indexing set. As above, by applying the Hurewicz theorem(Hurewicz 1955) we observe that

$$\Pi_{n+k}(\bigvee_{u \in c} S_u^{n+k}) \cong H_{n+k}(\coprod_c S^{n+k}) \cong F_c.$$

Let

$$f: \bigvee_c S^{n+k} \to X_{n+k},$$

be a map which is restricted to the sphere S_u^{n+k} representing the generator $\partial u \in \Pi = \Pi_{n+k}(X_{n+k}).$

We define X_{n+k+1} to be mapping cone of $f:$

$$X_{n+k+1} = X_{n+k} u_f \bigcup_{u \in c} D^{n+k+1}.$$

By theorem 4.1 we have that

$$\Pi_q(X_{n+k}) \to \Pi_q(X_{n+k+1}),$$

is an isomorphism for $q < n+k,$ and

$$\Pi_{n+k}(X_{n+k}) \to \Pi_{n+k}(X_{n+k-1}),$$

is surjective, with kernel the subgroup generated by $\{\partial u : u \in c\},$ since this subgroup generates equation (4.2). We see that the homomorphism is zero. Since it is surjective, implies that

$$\Pi_{n+k}(X_{n+k+1}) = 0.$$

Hence X_{n+k+1} has the required property on its homotopy groups.

5.0 CONCLUSION

This paper has shown that the compact open function space topology is a fibration which satisfies the homotopy lifting property (HLP). Grassmannian models were applied to show that the sphere bundles are isomorphic to the unit sphere bundle of the universal vector bundle. Since the homomorphism is zero implies that $\prod_{n+k}(X_{n+k+1}) = 0.$ Hence X_{n+k+1} has the required property on its homotopy groups.

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