REMARKS ON NONLOCAL SYMMETRIES AND ERMANNO-BERNOULLI CONSTANTS FOR REDUCING SOME DYNAMICAL SYSTEMS

Arunaye, F.I.
Department of Mathematics and Computer Science
Delta State University, Abraka
E-mail: ifarunaye@yahoo.com ; festus.arunaye@uwimona.edu.jm

ABSTRACT

In the consideration of nonlocal symmetries of the dynamical system $\ddot{x} + gr^{-1}x + hr^{-1}\theta = 0$, it was shown by Leach and Flessas (2003), using the Ermanno-Bernoulli reduction process that the radial reduction variable $u_1 = J\cos y$ defined the harmonic oscillator of its reduced system of equations. However we observed that this variable may be misleading and had affected the corresponding nonlocal symmetries presented in Leach and Flessas (2003). In this remark, we present the correct radial reduction variable and the corresponding nonlocal symmetries for the system. We also compute nonlocal symmetries of a related dynamical system $\ddot{x} + Px + L^{-1}g\dot{x} = 0$ and note that there is a close relationship between these two dynamical systems from their symmetries.

Keywords: Nonlocal symmetry, Reduction process, Laplace-Runge-Lenz vectors, Ermanno-Bernoulli constants, Dynamical systems.

INTRODUCTION

Scientific interests in the analysis of second order differential equations (in particular dynamical systems) due to their manifestations in various physical, biological, geometry, fluid dynamics and host of other mathematical sciences cannot be over emphasized. It is paramount to note that solutions to many of these second order equations are still being sort, since in some instances there are no definite methods for obtaining them and in other cases no optimum solutions are seemingly attainable. The work of Lie (1881, 1895) and the works of (Ovsiannikov, 1982; Olver, 1986, 2003; Gorringe and Leach, 1987; Stephani, 1989; Bluman and Cole, 1974; Bluman and Kumei, 1989; Bluman and Anco, 2002; Andriopoulos *et al.,* 2002; Leach and Flessas, 2003; Leach and Nucci, 2004; Leach *et al.,* 2003; Arunaye and White, 2007; Arunaye, 2009) in symmetry transformations of differential equations, however brought to the fore the systematic method of obtaining solutions to many simple and complicated equations (be it ordinary or partial). In recent time nonlocal symmetry analysis is prominent in the literature, particularly the fact that there exists at least one integrable second order equation with no Lie point symmetry but possesses nonlocal symmetries; consequently integrability of equations relied on the existence of nonlocal symmetries (Arunaye, 2009; Leach and Andriopoulos, 2007).

ON THE NATURAL REDUCTION VARIABLES FOR DYNAMICAL SYSTEM

Given the dynamical system

$$\ddot{x} + gr^{-1}x + hr^{-1}\theta = 0, \quad |x| = r; \qquad (1)$$

Where $g = \dfrac{u''(\theta) + u(\theta)}{r^2} + \dfrac{2v'(\theta)}{r^{3/2}}$ and $h = \dfrac{v(\theta)}{r^{3/2}}$, it was shown by Leach and Flessas (2003) that its Laplace-Runge-Lenz vector is

$$J = \dot{r} \wedge L - ue_r - [u' + 2r^{1/2}v]e_\theta, \qquad (2)$$

where $\wedge$ is vector product, L is the angular momentum of motion and e_r, e_θ unit vectors in direction of motion.

The Ermanno-Bernoulli constants with which (1) were reduced by Leach and Flessas (2003) is as follows:

$$J_{\pm} = -J_1 \pm iJ_2 = [(\frac{L^2}{r} - u) \pm i(\frac{L^2}{r} - u)']e^{\pm i\theta}$$

$$= (u_1 \pm iu_1')e^{\pm i\theta} \qquad (3)$$

where $u_1 = \frac{L^2}{r} - u$, (4)

And $0 = \frac{L'}{L^2} + \frac{v(\theta)}{(u_1 + u(\theta))^{3/2}}$. (5)

We note that equation (3) implies

$$u_1 = \tfrac{1}{2}(J_+ e^{-i\theta} + J_- e^{+i\theta}) = J\cos y \quad , \qquad (6)$$

where $J = |J_+| = |J_-|$ and equation (5) implies

$$u_2 = \frac{1}{L} - \int \frac{v(\theta)d\theta}{(J\cos\theta + u(\theta))^{3/2}} \qquad (7)$$

which both reduced (1) to the system of equations

$$u_1'' + u_1 = 0 \quad , \quad u_2' = 0 \qquad (8)$$

ON THE NONLOCAL SYMMETRY OF SYSTEM (1)

The Lie point symmetry analysis of system of equations (8) is well known. While the corresponding nonlocal symmetries as presented by Ref. Leach and Flessas (2003) are contentious due to the unfortunate radial reduction variable $u_1 = J\cos y$. Thus the equation (5.7.29) of Leach and Flessas (2003) is misleading, and hence the corresponding symmetries are not well posed.

Considering the Ermanno-Bernoulli constant J_+ ; the following relations are true:

$$J_+ = J_1 + iJ_2 = (u_1 + iu_1')(\cos\theta + i\sin\theta)$$

$$= u_1\cos\theta - u_1'\sin\theta + i(u_1\sin\theta + u_1'\cos\theta) \quad . \qquad (9)$$

i.e. $J_1 = u_1\cos\theta - u_1'\sin\theta$,

$$J_2 = u_1\sin\theta + u_1'\cos\theta \quad . \qquad (10)$$

From (10) we have

$$u_1 = J_1\cos\theta + J_2\sin\theta \quad , \qquad (11)$$

$$u_1' = -J_1\sin\theta + J_2\cos\theta \quad ;$$

showing that $u_1 \neq J\cos y$ contrary to known variable of Leach and Flessas (2003), but equality holds provided $J = |J_1| = |J_2|$. Further, it is clear that (11) implies

$$u_1 = J\tfrac{1}{2}(e^{i\delta}e^{-i\theta} + e^{-i\delta}e^{i\theta}) = J\cos(\theta - \delta) \qquad (12)$$

where δ is some non trivial function of J_1 and J_2 . Thus

$$J_+ = Je^{i\delta} \; ; \; J_- = Je^{-i\delta} \quad . \qquad (13)$$

It is now shown that the following are the appropriate natural variables for reducing (1) to (8)

$$u_1 = J\cos(\theta - \delta) \quad ,$$

$$u_2 = \frac{1}{L} - \int \frac{v(\theta)d\theta}{(J\cos(\theta - \delta) + u(\theta))^{3/2}} \qquad (14)$$

While the appropriate corresponding nonlocal symmetries are obtained as follows:

$$\Gamma_1 = 3[\int r^2\dot{\theta}dt]\partial_t + 2r^3\dot{\theta}\partial_r \quad ,$$

$$\Gamma_2 = [\int \left(\tfrac{2u'}{r^2\dot{\theta}^2} + \tfrac{3vr^2\dot{\theta}}{u + J\cos(\theta-\delta)}\right)dt]\partial_t + [\tfrac{u'}{r^2\dot{\theta}^2} + \tfrac{2vr^3\dot{\theta}}{u + J\cos(\theta-\delta)}]\partial_r - \partial_\theta \quad ,$$

$$\Gamma_3 = [t - \int \tfrac{udt}{r^3\dot{\theta}^2}]\partial_t + [r - \tfrac{u}{r^2\dot{\theta}^2}]\partial_r \quad ,$$

$$\Gamma_{4\pm} = 2[\int \frac{e^{\pm i\theta}}{r^3\dot{\theta}^2}dt]\partial_t + \frac{e^{\pm i\theta}}{r^2\dot{\theta}^2}\partial_r \quad , \qquad (15)$$

$$\Gamma_{6\pm} = [\int e^{\pm 2i\theta}\left(\tfrac{3vr^2\dot{\theta}}{u + J\cos(\theta-\delta)} + \tfrac{2(u' \mp iu)}{r^3\dot{\theta}^2}\right)dt]\partial_t - [e^{\pm 2i\theta}\left(\tfrac{2vr^3\dot{\theta}}{u + J\cos(\theta-\delta)} \pm ir + \tfrac{u' \mp iu}{r^3\dot{\theta}^2}\right)]\partial_r + e^{\pm 2i\theta}\partial_\theta \quad ,$$

$$\Gamma_{8\pm} = [\int e^{\pm i\theta}\left(2u'(1 - \tfrac{u}{r^3\dot{\theta}^2}) - 3(r^2\dot{r}\dot{\theta} \pm ir^3\dot{\theta}^2 - 2r^{1/2}v \mp iu\right)dt]\partial_t$$

$$-[e^{\pm i\theta}\left(\tfrac{2vr^2\dot{\theta}}{u + J\cos(\theta-\delta)}(u' \pm i(r^3\dot{\theta}^2 - u))(1 - \tfrac{u}{r^3\dot{\theta}^2})\right)]\partial_r + e^{\pm i\theta}(r^3\dot{\theta} - u)\partial_\theta \quad .$$

A RELATED DYNAMICAL SYSTEM

We consider the dynamical system

$$\ddot{x} + px + L^{-1}g\dot{x} = 0 \quad , \qquad (16)$$

where L is the angular momentum of motion and $\dot{L} = -g$ since the angular momentum is geometrically non-constant (Leach and Flessas, 2003; Andriopoulos *et al.,* 2002), that is

$$x \wedge \ddot{x} + L^{-1}gx \wedge \dot{x} = 0 \quad , \qquad (17)$$

the symbol $\wedge$ is vector product .
The Laplace-Runge-Lenz vector is obtained from the relation

$$L \wedge \ddot{x} + PL \wedge x + L^{-1} gL \wedge \dot{x} = 0 , \qquad (18)$$

which is (functionally)

$$J = L \wedge \dot{x} - u(r,\theta)e_r - v(r,\theta)e_\theta , \qquad (19)$$

Provided

$$\frac{d}{dt}(-ue_r - ve_\theta) = -(pr^3\dot{\theta} - g\dot{r})e_\theta - 2gr\dot{\theta}e_r . \qquad (20)$$

Equation (20) produced the following:

$$v\dot{\theta} - \dot{u} = -2gr\dot{\theta} , \quad \text{(i)} \qquad (21)$$

$$u\dot{\theta} + \dot{v} = pr^3\dot{\theta} - g\dot{r} . \quad \text{(ii)}$$

Equation (i) of (21) implies

$$(v - u_\theta)\dot{\theta} - \dot{r}u_r = -2gr\dot{\theta} ,$$

i.e. $u_r = 0 \quad \Rightarrow \quad u = u(\theta)$, (22)

$$v - u_\theta = -2gr \quad \Rightarrow \quad v = u'(\theta) - 2gr . \qquad (23)$$

Equation (ii) of (21) implies

$$(u + v_\theta)\dot{\theta} + \dot{r}v_r = pr^3\dot{\theta} - g\dot{r} ,$$

i.e. $v_r = -g$; $u + v_\theta = pr^3$.

Taking $g = r^{-3/2}v(\theta)$ (24)
we have

$$v = u'(\theta) - 2r^{-1/2}v(\theta) . \qquad (25)$$

i.e. $u(\theta) + u''(\theta) - 2r^{-1/2}v'(\theta) = pr^3$,

so, $$p = \frac{1}{r^3}[u(\theta) + u''(\theta) - 2r^{-1/2}v'(\theta)] . \qquad (26)$$

Substituting for (22), (24), (25) and (26) in (16) we obtain a specific dynamical system

$$\ddot{x} + \frac{1}{r^2}[u''(\theta) + u(\theta) - 2r^{-1/2}v'(\theta)]e_r + \frac{v(\theta)}{r^{5/2}}e_\theta ;$$

(27)
whose conserved vector is

$$J = L \wedge \dot{x} - u(\theta)e_r - [u'(\theta) - 2r^{-1/2}v(\theta)]e_\theta .$$

(28)

RELATED HAMILTONIAN SYSTEM

The related Hamiltonian system (Leach and Flessas 2003) is given by

$$H = \tfrac{1}{2}p \cdot p + W(r,\theta) , \qquad (29)$$

from which the Newtonian equation of motion is calculated as

$$\ddot{r} = -\nabla W(r,\theta) = F(r,\theta) , \qquad (30)$$

and in a plane polar coordinates we have

$$\nabla W(r,\theta) = \frac{\partial W}{\partial r}e_r + \tfrac{1}{r}\frac{\partial W}{\partial \theta}e_\theta . \qquad (31)$$

So

$$\frac{\partial W}{\partial r} = r^{-2}[u''(\theta) + u(\theta)] - 2r^{-5/2}v'(\theta) ,$$

$$\frac{\partial W}{\partial \theta} = r^{-3/2}v(\theta) . \qquad (32)$$

The consistent solution of system (32) places considerable constraints on the functions $u(\theta)$ and $v(\theta)$ to obtain the Hamiltonian (Leach and Flessas, 2003; Sen, 1987; Gorringe and Leach, 1987). The constraints are

$$u''' + u' = 0 ,$$

$$4v'' + v = 0 . \qquad (33)$$

SYMMETRY ANALYSIS OF SYSTEM (16)

We present the nonlocal symmetries of the dynamical system (16) as follows.
The Ermanno-Bernoulli constants

$$J_\pm = -J_1 \pm iJ_2 = [(\frac{L^2}{r} + u) + i(\frac{L^2}{r} + u)']e^{\pm i\theta}$$

reduced system (16) to system

$$u_1'' + u_1 = 0 ,$$

$$u_2' = 0 ;$$

where

$$J = -(\frac{L^2}{r} + u(\theta))e_r + (\dot{r} - u'(\theta) + 2r^{-1/2}v(\theta))e_\theta ,$$

$$e_r = \hat{i}\cos\theta + \hat{j}\sin\theta ; \quad e_\theta = -\hat{i}\sin\theta + \hat{j}\cos\theta ,$$

and $u_1 = J\cos(\theta - \delta)$,

$$u_2 = \frac{1}{L} - \int \frac{v(\theta)d\theta}{(J\cos(\theta - \delta) + u(\theta))^{3/2}} .$$

We obtain the nonlocal symmetries of (16) as follow:

$$\Gamma_{*1} = 3[-\int r\dot{\theta}dt]\partial_t + r\dot{\theta}\partial_r ,$$

$$
\begin{aligned}
\Gamma_{*2} &= [-\int\left(\tfrac{2u'}{r\dot{\theta}^2} + \tfrac{3vr\dot{\theta}}{J\cos(\theta-\delta)-u}\right)dt]\partial_t + [\tfrac{u'}{2r^2\dot{\theta}^2} + \tfrac{vr\dot{\theta}}{J\cos(\theta-\delta)-u}]\partial_r - \partial_\theta, \\
\Gamma_{*3} &= [t + \int \tfrac{udt}{r^3\dot{\theta}^2}]\partial_t + [\tfrac{1}{2r} - \tfrac{u}{2r^2\dot{\theta}^2}]\partial_r, \\
\Gamma_{*4\pm} &= 2[-\int \frac{e^{\pm i\theta}}{r^2\dot{\theta}^2} dt]\partial_t + \frac{e^{\pm i\theta}}{2r^2\dot{\theta}^2}\partial_r, \qquad (34) \\
\Gamma_{*6\pm} &= [-\int e^{\pm 2i\theta}\left(\tfrac{3vr\dot{\theta}}{J\cos(\theta-\delta)-u} + \tfrac{2(u'\pm iu)}{r^3\dot{\theta}^2}\right)dt]\partial_t \\
&\quad -[e^{\pm 2i\theta}\left(\tfrac{vr^3\dot{\theta}}{J\cos(\theta-\delta)-u} \pm \tfrac{i}{2r} + \tfrac{u'\mp iu}{2r^2\dot{\theta}^2}\right)]\partial_r + e^{\pm 2i\theta}\partial_\theta, \\
\Gamma_{*8\pm} &= [\int e^{\pm i\theta}\left(2u'(1+\tfrac{u}{r^2\dot{\theta}^2}) + 3(r\dot{r}\dot{\theta} \mp ir^2\dot{\theta}^2 + 2r^{-1/2}v \mp iu\right)dt]\partial_t \\
&\quad -[e^{\pm i\theta}\left(\tfrac{v\dot{\theta}}{J\cos(\theta-\delta)-u}(-u' \mp i(r^2\dot{\theta}^2 - u))(1+\tfrac{u}{r^2\dot{\theta}^2})\right)]\partial_r \\
&\quad - e^{\pm i\theta}(r^2\dot{\theta}^2 + u)\partial_\theta .
\end{aligned}
$$

Comparing the nonlocal symmetries of dynamical systems (1) and (16) obtained respectively by the use of the natural reduction variables obtained from the Laplace-Runge-Lenz vectors (2) and (28) we observed that systems (1) and (16) are equivalence dynamical systems; that is one is map onto the other by a point transformation.

CONCLUSION

The importance of Lie symmetry method for obtaining the solutions of differential equations as well as the significance of nonlocal symmetries to integrable differential equations were referenced in the introductory section of this work. In the second part we presented the correct natural reduction variables of system (1), and hence the corresponding nonlocal symmetries. Finally we consider a related dynamical system (16) and by the use of its natural reduction variable, usually from its Ermanno-Bernoulli constants we obtained its nonlocal symmetries which are equivalence to those of system (1).

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