CHARACTERIZATIONS OF REAL-VALUED SINGLE VARIABLE QUASICONCAVE FUNCTIONS

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ABSTRACT

This paper investigates quasiconcave functions. It shows that real-valued quasiconcave functions of single variable can be characterized through knowledge of the minimum value of the function with respect to some points of interest in a given interval. This is a variant of Jensen's inequality for quasiconcave functions. We also shows that quasiconcave functions can be characterized through a comparison of function values; the line segment minimum property; the upper level set and the derivative, precisely the first and second order conditions.

Keywords: Quasiconcavity, Level Sets, Line Segment Minimum Property, Twice Differentiable Function, Inequality.

1.0 INTRODUCTION

Most times it is assumed that analyses on $\mathbb{R}^n$ are more acceptable with regards to application, and further that they generalize those on $\mathbb{R}$. Unfortunately these assumptions leave new entrants into the field of mathematics handicapped and perhaps into concluding that the mountains are unscalable! In addition, it is a known fact that the results and proofs for works on $\mathbb{R}^n$ and $\mathbb{R}$ do not always follow the same trend. This work presents a strong base for the understanding of quasiconcave functions on $\mathbb{R}.$ It presents characterizations of real-valued single-variable quasiconcave functions. These results have not been proved on $\mathbb{R}.$

A quasiconcave function is a real-valued function defined on an interval or a convex subset of a real vector space such that the inverse image of any set of the form $(-\infty, a)$ is a convex set. As will be seen later all concave functions are quasiconcave, but not all quasiconcave functions are concave. So quasiconcavity is a generalization of concavity.

Definition 1.1 *If* $x, y \in \mathbb{R}$ *and* $\lambda \in [0,1],$ *then* $\lambda x + (1-\lambda)y$ *is a convex combination of* x *and* y. Geometrically, a convex combination of x and y is a point somewhere between x and y (Pemberton and Nicholas, 2007).

Definition 1.2 *A set* $I \subseteq \mathbb{R}$ *is convex if* $x, y \in I$ *implies* $\lambda x - (1-\lambda)y \in I$ *for all* $\lambda \in (0,1).$ The definition of a convex set immediately implies that I is convex if and only if I is either empty, a point, or an interval. Throughout this work we suppose that I is a convex subset of $\mathbb{R}$ (Pemberton and Nicholas, 2007).

Definition 1.3 $f: I \longrightarrow \mathbb{R}$ *is concave if for any* $x, y \in I$*, we have, for all* $\lambda \in [0,1],$

$$f(\lambda y - (1-\lambda)x) \geq \lambda f(y) - (1-\lambda)f(x). \quad (1)$$

$f: I \longrightarrow \mathbb{R}$ *is strictly concave if for any* $x, y \in I$*, with* $x \neq y$*, we have, for all* $\lambda \in [0,1],$

$$f(\lambda y - (1-\lambda)x) > \lambda f(y) - (1-\lambda)f(x).$$

(Pemberton and Nicholas, 2007; Goyan and Ravi,2008; Simon, 2011).

In words, a function is concave if its value at the linear combination between two points in its domain is greater than or equal to the weighted average of the function's values at each of the points considered

In practical terms, the difference is that concavity allows for linear segments, but strict concavity does not. Concavity allows for ascending and descending linear segments. Vertical segments are excluded because of $x \neq y$. Horizontal segments are excluded, because such lines would allow chords to be drawn above the curve, violating the requirements of equation (Pemberton and Nicholas, 2007).

2.0 QUASICONCAVE FUNCTIONS

Definition 2.1

A function $f: I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ *is called quasiconcave if its domain and all its superlevel sets*

$$I_\beta = \{x \in dom\ f | f(x) \geq \beta\}, \quad (3)$$

for $\beta \in \mathbb{R}$ *are convex* (Goyal and Ravi, 2008; Boyd and Vandenberg, 2004).

A function is *quasiconvex* if $-f$ is *quasiconcave*, that is, every *sublevel set* $\{x \in dom\ f | f(x) \leq \beta\}$ is convex. A function that is both quasiconcave and quasiconvex is called quasilinear. A function is quasilinear if its domain, and every level set $\{x | f(x) = \beta\}$ is convex.

Quasiconcavity requires that each sublevel set be an interval (including, possibly, an infinite interval).

2.2 Examples of Quasiconcave Functions

The following are examples of quasiconcave functions on $\mathbb{R}$:

Logarithm: $\log x$ *on* $\mathbb{R}_{--}$ *is quasiconcave (and quasiconvex, hence quasilinear)*

Ceiling function: $ceil(x) = inf\ \{z \in Z | z \leq x\}$ *is quasiconcave (and quasiconvex)*

These examples show that quasiconcave functions can be discontinuous (Boyd and Vandenberg, 2004).

We can give a simple characterization of quasiconcave functions on $\mathbb{R}$. We consider continuous functions, since stating the conditions in the general case is cumbersome:

A continuous function $f: \mathbb{R} \longrightarrow \mathbb{R}$ *is quasiconcave if, and only if, at least one of the following conditions holds:*

f *is nonincreasing*

f *is nondecreasing*

there is a point $c \in dom\ f$ *such that for* $t \leq c$ *(and* $t \in dom\ f$*),* f *is nondecreasing, and* $t \geq c$ *(and* $t \in dom\ f$*), and* f *is nonincreasing (*Boyd and Vandenberg, 2004).

3. 0 THE LINE SEGMENT AND LOCAL MINIMUM PROPERTY

It is quite difficult to get simple necessary and sufficient conditions for quasiconcavity in a case where f is twice continuously differentiable. Thus we will need the following definitions

Definition 3.1 *Let* $I \subseteq \mathbb{R}$ *be a nonempty open interval, then* $f: I \rightarrow \mathbb{R}$ *has the line segment minimum property if and only if for* $x, y \in I$, $x \neq y$,

$$min_\alpha \{f(\alpha x - (1-\alpha)y): 0 \leq \alpha \leq 1\} \quad (3)$$

exists (Diewert *et al.*, 1981).

That is, the minimum of f along any line segment in its domain of definition exists.

It is easy to verify that if f is a quasiconcave function defined over the interval I, then it satisfies the line segment minimum property (3), since the minimum will be attained at one or both of the endpoints of the interval; that is, the minimum will be attained at either $f(x)$ or $f(y)$ (or both points) since $f(\alpha x+(1-\alpha)y)$ for $0 \le \alpha \le 1$ is equal to or greater than $\min\{f(x), f(y)\}$ and this minimum is attained at either $f(x)$ or $f(y)$ (or both points).

Definition 3.2 *Let* $\alpha \in \mathbb{R}$, *the function* h *defined over an interval* I' *attains a semistrict minimum at* $t_0 \in Int\ I'$ *if and only if there exist* $\varepsilon_1 > 0$ *and* $\varepsilon_2 > 0$ *such that* $\alpha_0 - \varepsilon_1$, $\alpha_0 - \varepsilon_2 \in I$ *and*

$$h(\alpha_0) \le h(\alpha) \quad (4)$$

for all α *such that* $\alpha \in [\alpha_0 - \varepsilon_1,\ \alpha_0 - \varepsilon_2]$;

$$h(\alpha_0) < h(\alpha_0 - \varepsilon_1) \quad (5)$$

and

$$h(\alpha_0) < h(\alpha_0 - \varepsilon_2) \quad (6)$$

(Goyal And Ravi, 2008).

If h just satisfies (4) at the point α_0, then it can be seen that it attains a *local minimum* at α_0. But conditions (5) and (6) show that a *semistrict local minimum* is stronger than local minimum: for h to attain a semistrict local minimum at such α_0, we need h to attain a local minimum at such α_0, but the function must eventually strictly increase at the end points of the region where the function attains the local minimum. Note that h attains a *strict local minimum* at such $\alpha_0 \in Int\ I'$ if and only if there exists $\varepsilon > 0$ such that $\alpha_0 - \varepsilon,\ \alpha_0 - \varepsilon \in I'$ and

$$h(\alpha_0) < h(\alpha) \quad (7)$$

for all α such that $\alpha_0 - \varepsilon \le \alpha \le \alpha_0 - \varepsilon$ but $\alpha \ne \alpha_0$.

It can be seen that if h attains a strict local minimum at α_0, then it also attains a semistrict local minimum at α_0. Hence, a semistrict local minimum is a concept that is intermediate to the concept of a local and strict local minimum (Bronson and Naadimuthu, 1997; Peresini *et al.,* 1993; Wenyu and Yaxiang, 2006).

4.0 CHARACTERIZATIONS OF QUASICONCAVE FUNCTIONS

Theorem 4.1 The Minimum Function Value Test Characterization of Quasiconcave Functions: *Let* $I \subseteq \mathbb{R}$ *be a closed interval, a function* $f : I \to \mathbb{R}$ *is quasiconcave if and only if for* $x, y \in I$ *and* $\lambda \in [0,1]$

$$f(\lambda x - (1-\lambda)y) \ge \min\ \{f(x),\ f(y)\}. \quad (8)$$

Proof: First, let $I_\beta = \{x \in dom\ f | f(x) \ge \beta\}$ be convex. Take $x, y \in I$ and $\lambda \in [0,1]$. Assume without loss of generality that

$$f(x) \ge f(y) = \beta. \quad (9)$$

Thus, $x, y \in I_\beta$, and by the convexity of I_β we have that $\lambda x - (1-\lambda)y \in I_\beta$, which means that

$$f(\lambda x - (1-\lambda)y) \ge \beta = f(y) = \min\ \{f(x),\ f(y)\}. \quad (10)$$

Suppose $f(\lambda x - (1-\lambda)y) \ge \min\ \{f(x),\ f(y)\}$ for all $x, y \in I$ and $\lambda \in [0,1]$. Let $x, y \in I_\beta$, then $f(x) \ge \beta$ and $f(y) \ge \beta$, and so $\min\{f(x),\ f(y)\} \ge \beta$. By hypothesis

$f(\lambda x - (1-\lambda)y) \geq \min\ \{f(x),\ f(y)\}$ and hence $\lambda x - (1-\lambda)y \in I_g$. ■

The above result means that the line segment joining x to y that has height equal to the minimum value of the function at the point x and y lie below (or is coincident with) the graph of f along the line segment joining x to y (Frenchel, 1953; Arrow and Enthoven, 1961; Avriel *et al.*, 1988; Ginsberg, 1974; Ortega and Rheinboldt, 1970). This is a variant of Jensen's inequality that characterizes quasiconcavity.

If f is concave over I, then

$f(\lambda x - (1-\lambda)y) \geq \lambda f(x) - (1-\lambda)f(y)$ (by (2)11)

$\geq \min\ \{f(x), f(y)\}$, (12)

where the second inequality results from the fact that $\lambda f(x) - (1-\lambda)f(y)$ is an average of $f(x)$ and $f(y)$. Thus if f is concave it is also quasiconcave. This characterization can be written in an equivalent form as shown in the next result, thus we can use them interchangeably.

Theorem 4.2 The Function Value Comparison Characterization of Quasiconcavity

Let $\emptyset \neq I \subseteq \mathbb{R}$ *be open, then* $f: I \to \mathbb{R}$ *is quasiconcave if and only if for* $x, y \in I$, $x \neq y$

$$f(y) \geq f(x) \Rightarrow f(\lambda x - (1-\lambda)y) \geq f(x) \quad (13)$$

Proof: Suppose

$f(y) \geq f(x) \Rightarrow f(\lambda x - (1-\lambda)y) \geq f(x)$.

Since $\lambda x - (1-\lambda)y \in [x, y] \subseteq I$, it follows that

$$f(\lambda x - (1-\lambda)y) \geq f(x) = \min\{f(x), f(y)\}. \quad (14)$$

Conversely, suppose

$f(\lambda x - (1-\lambda)y) \geq \min\{f(x), f(y)\}$,

then for $f(y) \geq f(x)$ we have that

$$\min\{f(x), f(y)\} = f(x). \quad (15)$$

Thus

$f(\lambda x - (1-\lambda)y) \geq \min\{f(x), f(y)\} = f(x)$. ■

This means that the function f is quasiconcave if $f(y) \geq f(x)$ implies that its value at a convex combination of two points in its domain is greater than or equal to $f(x)$ which is the function value of the smaller function value of the two points. A version of this result for functions defined on $\mathbb{R}^n$ exists in Mangasarian (1969 and Takayama (1995).

4.3 The Derivative-Based Characterization of Quasiconcavity

Theorem 4.3a First Order Condition:

Let $f: I \to \mathbb{R}$ *be a once differentiable function defined over the open interval* $I \subseteq \mathbb{R}$, $x, y \in I$, $x \neq y$; *then* f *is quasiconcave if and only if*

$$f'(x)(y-x) < 0 \Rightarrow f(y) < f(x). \quad (16)$$

Proof: (8) $\Rightarrow$ (16). We show that not (16) implies not (8). Not (16) means that there exist $x, y \in I$, $x \neq y$ such that

$$f'(x)(y-x) \quad (17)$$

and

$$f(y) \geq f(x). \quad (18)$$

Define the function of one variable h for $0 \leq \alpha \leq 1$ by

$$h(\alpha) = f(x - \alpha(y-x)). \quad (19)$$

It can be verified that

$$h(0) = f(x) \quad and \quad g(1) = f(y). \quad (20)$$

It can also be verified that the derivative of $h(\alpha)$ for $0 \leq \alpha \leq 1$ can be computed as follows

$$h'(\alpha) = f'(x - \alpha(y - x))(y - x). \quad (21)$$

Evaluating (21) at $\alpha = 0$ and using (17) shows that

$$h'(0) < 0. \quad (22)$$

Since the first order partial derivative of f is continuous, it can be seen that (22) implies the existence of a ε such that

$$0 < \varepsilon < 1 \quad (23)$$

and

$$h'(\alpha) < 0 \quad (24)$$

for all α such that $0 \leq \alpha \leq \varepsilon$.

Thus $h(\alpha)$ is a decreasing function over this interval of α's and thus

$$h(\varepsilon) = f(x + \varepsilon(y - x)) = f((1 - \varepsilon)x + \varepsilon y)) < h(0) = f(x). \quad (25)$$

But (23) and (25) imply that

$$f(\varepsilon(x + (1 - \varepsilon)y) < f(x) \quad (26)$$

where $\lambda \equiv 1 - \varepsilon$. Since (23) implies $0 < \lambda < 1$, (26) contradicts (8).

(16) $\Longrightarrow$ (8). We show not (8) implies not (16). Not (8) means that there exist $x, y \in I,\ x \neq y$ and $\lambda^* \in (0,1)$ such that

$$f(x) \leq f(y) \quad (27)$$

and

$$f(\lambda^* x + (1 - \lambda^*)y) < f(x). \quad (28)$$

For $\alpha \in [0,1]$, define the function $h(\alpha)$ as follows:

$$h(\alpha) \equiv f(x + \alpha[y - x]). \quad (29)$$

Define α^* as follows: $\alpha^* \equiv 1 - \lambda^*$ (30)

and note that $\alpha^* \in (0, 1)$ and

$$g(\alpha^*) = f(\lambda^* x + (1 - \lambda^*)y) \quad (31)$$

$< f(x)$ (By (28))

$= h(0)$ (By (29))

The continuity of f' implies that $h'(\alpha)$ and $h(\alpha)$ are continuous functions of $\alpha \in [0,1]$.

Now consider $h(\alpha)$ along the line segment $0 \leq \alpha \leq \alpha^*$. The inequality (31) shows that $h(\alpha)$ eventually decreases from $h(0)$ to the lower number $h(\alpha^*)$ along this interval. Thus there must exist a α^{**} such that

$$0 \leq \alpha^{**} < \alpha^*; \quad (32)$$

$$h(\alpha) \leq h(0) \quad (33)$$

for all $\alpha \in [\alpha^{**}, \alpha^*]$ and

$$h(\alpha^{**}) = h(0). \quad (34)$$

Essentially, the (32), (33) and (34) say that there exists a close interval to the immediate left of the point α^*, $[\alpha^{**}, \alpha^*]$, such that $h(\alpha)$ is less than or equal to $h(0)$ for α in this interval and the lower boundary point of the interval, α^{**}, is such that $h(\alpha^{**})$ equals $h(0)$.

$$h'(\alpha) \geq 0 \quad (35)$$

for α such that $\alpha \in [\alpha^{**}, \alpha^*]$. Then by the Mean Value Theorem, there exists α^{***} such that $\alpha^{**} < \alpha^{***} < \alpha^*$ and

$$h(\alpha^*) = h(\alpha^{**}) + h'(\alpha^{***})(\alpha^* - \alpha^{**}) \quad (36)$$

$\geq h(\alpha^{**})$ (By (35))

$= h(0)$ (By (34))

But (36) contradicts $h(\alpha^*) < h(0)$, which is equivalent (28). Thus our supposition is false.

Hence there exists α' such that

$$\alpha^{**} < \alpha' < \alpha^{*} \qquad (37)$$

and

$$h'(\alpha') < 0 \qquad (38)$$

By (33) we have

$$h(\alpha') = h(0) \qquad (39)$$

Using definition (29), the inequalities (38) and (39) translate into the following inequalities:

$$h'(\alpha') = f'\big(x + \alpha'(y - x)\big) < 0; \qquad (40)$$

$$h(\alpha') = f(x + \alpha'(y - x)) \leq f(x) \quad \text{(By (27)) 41)}$$

$$\leq f(y) \qquad (42)$$

Now define

$$z \equiv x + \alpha'(y - x) \qquad (43)$$

and note that the inequalities (37) imply that

$$0 < \alpha' < 1 \qquad (44)$$

Using definition (43), we have

$$y - z = y - (x + \alpha'(y - x)) \qquad (45)$$

$$= (1 - \alpha')(y - x) \quad = 0 \qquad (46)$$

Note that (46) implies that

$$y - x = (1 - \alpha')^{-1}(y - z) \qquad (47)$$

Substituting (43) and (47) into (41) we have that

$$(1 - \alpha')^{-1} f'(z)(y - z) < 0 \qquad (48)$$

$f'(z)(y - z) < 0$, since $(1 - \alpha')^{-1} > 0$

$$f(z) \leq f(y) \qquad (49)$$

The inequalities (48) and (49) show that (16) does not hold, with z playing the role of x in condition (16). ■

In this characterization we assumed that I is open and that the derivative of f exists and is a continuous function over I. This result is a single variable version of the several variable form on $\mathbb{R}^n$ as given in Mangasarian (1969) and Huang and Crook (1997).

Now consider the following result which is contrapositive to Theorem 4.3a making them (Theorem 4.3a and Corollary 4.3b) logically equivalent.

Corollary 4.3b First Order Condition: *Let $I \subseteq \mathbb{R}$ be an open interval in $\mathbb{R}$ and suppose $f: I \to \mathbb{R}$ is a once differentiable function, then f is quasiconcave if and only if*

$$f(y) \geq f(x) \Rightarrow f'(x)(y - x) \geq 0, \quad x, y \in I, \ x \neq y. \qquad (50)$$

(Arrow and Enthoven, (1961).

Proof: Condition (50) is contrapositive to condition (16) and is logically equivalent to it. ■

Next we present a characterization of quasiconcavity for functions defined on $\mathbb{R}$ through the line segment minimum property.

4.4 Line Segment Minimum Property Characterization of Quasiconcavity

Theorem 4.4 Suppose $f: I \subseteq \mathbb{R} \to \mathbb{R}$ has the line segment minimum property for $x \in I$, then f is quasiconcave if and only if

$$x, y \in I, x \neq y \Rightarrow h(\alpha) \equiv f(x + \alpha(y - x)) \qquad (51)$$

does not attain a semistrict local minimum for any α such that $0 < \alpha < 1$.

Proof: Quasiconcavity $\Rightarrow$ (51): This equivalent to showing that not (51) implies not (8). Not (51) means there exists α^{*} such that $0 < \alpha^{*} < 1$ and $h(\alpha)$ attains a semistrict local minimum at α^*. This implies the existence of α_1 and α_2 such that

$$0 \le \alpha_1 < \alpha^* < \alpha_2 \le 1; \quad (52)$$

$$h(\alpha_1) > h(\alpha^*); \quad h(\alpha_2) > h(\alpha^*). \quad (53)$$

From the definition of h (53) implies that

$$f(x + \alpha^*(y - x)) < \min\{f(x + \alpha_1(y - x)), f(x + \alpha_2(y - x))\}. \quad 54)$$

But (52) can be used to show that the point $x + \alpha^*(y - x)$ is a linear combination of the points $x + \alpha_1(y - x)$ and $x + \alpha_2(y - x)$, and hence (54) contradicts the definition of quasiconcavity, (8). Hence f is not quasiconcavity.

(56) $\Longrightarrow$ quasiconcavity (8): This is equivalent to showing not (8) implies not (51). Suppose f is not quasiconcave. Then there exist $x, y \in I$, and λ^* such that $0 < \lambda^* < 1$ and

$$f(\lambda^* x + (1 - \lambda^*)y) < \min\{f(x),\ f(y)\}. \quad (55)$$

Define $h(\alpha) \equiv f(x + \alpha(y - x))$ for $0 \le \alpha \le 1$. Since f is assumed to satisfy the line segment minimum property, there exists a α^* such that $0 \le \alpha^* \le 1$ and

$$h(\alpha^*) = \min_\alpha\{h(\alpha): 0 \le \alpha \le 1\} \quad (56)$$

The definition of h and (55) shows α^* satisfies $0 < \alpha^* < 1$ and

$$f(x + \alpha^*(y - x)) = f((1 - \alpha^*)x + \alpha^* y) < \min\{f(x), f(y)\}. \quad (57)$$

Thus f attains a semistrict local minimum, which contradicts (51). ■

4.5 The Derivative-Based Characterization of Quasiconcavity-Second Order Condition

Theorem 4.5: *Let* $I \subseteq \mathbb{R}$ *be a nonempty open interval. Then* $f: I \to \mathbb{R}$ *a twice differentiable function is quasiconcave if and only if for* $x, y, z \in I,$ *with* $y \neq x$

$$\text{and } f'(z)(y-x) = 0 \Rightarrow (i)\ f''(z)(y-x)^2 < 0 \text{ or } (ii)\ f''(z)(y-x)^2 = 0 \quad h(\alpha) = f(z + \alpha(y-x)) \quad (58)$$

does not attain a semistrict local minimum at $\alpha = 0$.

Proof: We need to show that (58) is equivalent to (51) in the twice differentiable case. (51) is equivalent to the property that for $x, y, z \in I, y \neq x$ and

$$h(\alpha) \equiv f(z + \alpha(y - x)) \quad (59)$$

does not attain a semistrict local minimum at $\alpha = 0$.

Consider case (*i*). If this case occurs, then $h(\alpha)$ attains a strict local maximum at $\alpha = 0$ and hence cannot attain a semistrict local minimum at $\alpha = 0$. Hence, in the twice differentiable case (58) is equivalent to (59). ■

The above result is similar to the several variable form in Diewert *et at.* (1981).

4.6 Upper Level Set Characterization of Quasiconcave Functions

Theorem 4.6: *Let* $I \subseteq \mathbb{R}$ *be a nonempty open interval. The function* $f: I \to \mathbb{R}$ *is quasiconcave if and only if for every* $\beta \in Range\ f$ *the upper level set*

$$I_\beta = \{x \in dom\ f | f(x) \ge \beta\} \quad (60)$$

is convex.

Proof: Since (8) characterizes quasiconcavity, it is sufficient to show that it is equivalent (60)

$(8) \Longrightarrow (60)$: Let $\beta \in Range\ f$, $x, y \in L(\beta)$ and $\lambda \in (0,1)$,

$$\Longrightarrow \quad f(x) \ge \beta \text{ and } f(y) \ge \beta. \quad (61)$$

From (8), we have that

$$f(\lambda x + (1-\lambda)y) \geq \min \ \{f(x), f(y)\} \geq \beta. \qquad (62)$$

where the last inequality follows by using (61). But (62) shows that $\lambda x + (1-\lambda)y \in L(\beta)$ and thus $L(\beta)$ is a convex set.

$(60) \Longrightarrow (8)$: Let $x, y \in I$, $0 < \lambda < 1$ and let $\beta \equiv \min \ \{f(x), f(y)\}$. Thus $f(x) \geq \beta$ and hence, $x \in I_\beta$. Similarly, $f(y) \geq \beta$ and hence, $y \in I_\beta$. Since I_β is convex, we have that $\lambda x + (1-\lambda)y \in I_\beta$. Hence using the definition of I_β, we have

$$f(\lambda x + (1-\lambda)y) \geq \beta = \min \ \{f(x), f(y)\}. \quad \blacksquare \qquad (63)$$

5.0 QUASICONCAVITY AS A GENERALIZATION OF CONCAVITY.

Now, let us consider the relationship between quasiconcavity and concavity.

Theorem 5.1 *Let* $f: I \subseteq \mathbb{R} \to \mathbb{R}$ *be concave then* f *is quasiconcave.*

proof: Let $x, y \in I \subseteq \mathbb{R}$ and $\lambda \in (0,1)$. Since f is concave on I we have that

$$f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda) f(y) \qquad (64)$$

$$\geq \min \ \{f(x), f(y)\}, \qquad (65)$$

where the second inequality results from the fact that $\lambda f(x) + (1-\lambda) f(y)$ is an average of $f(x)$ and $f(y)$. Thus if f is concave it is also quasiconcave. $\blacksquare$

That is every concave function is quasiconcave. The converse of this is not true. We show this in the next result.

Corollary 5.2 *Not every quasiconcave function is concave*

Proof: To prove this, it is sufficient to show a counter-example. Now, define a function f on $\mathbb{R}_+$ by

$$f(x) = 3x^2 + 2x + 1.$$

We will first show that f is quasiconcave and thereafter show that it is not concave.

By definition f is an increasing function, and so $y \geq x \quad \Longrightarrow \quad f(y) \geq f(x)$ for any $\lambda \in (0,1)$. Thus $\lambda x + (1-\lambda)y \geq x.$

By the increasing nature of f we have that $f(\lambda x + (1-\lambda)y) \geq f(x)$
$\Longrightarrow \ f$ is quasiconcave.

Next we show that f is not concave. Consider two points $x, y \in \mathbb{R}_+$ where $x = 0 \text{ and } y = 3$, and $\lambda = 0.3$. Then

$$f(\lambda x + (1-\lambda)y) = f((0.3)(0) + (1-0.3)(3)) = f(2.1) = 18.43 \qquad (66)$$

Now
$f(0) = 1 \quad \text{and} \quad f(3) = 34$
so that

$$\lambda f(x) + (1-\lambda) f(y) = (0.3) f(3) + (0.7) f(0) = 3.7 \qquad (67)$$

which implies that
$f(\lambda x - (1-\lambda)y) < \lambda f(x) - (1-\lambda) f(y)$

Thus f is not concave. $\blacksquare$
Thus quasiconcavity is a generalization of concavity. By extension we have the following results for the relationship between convexity and quasiconvexity.

Theorem 5.3 *Let* $f: I \subseteq \mathbb{R} \to \mathbb{R}$ *be convex then* f *is quasiconvex.*

Corollary 5.4 *Not every quasiconvex function is convex.*

To proves of these last two results follow similar arguments as in the previous two. Thus quasiconvexity is a generalization of convexity.

6.0 CONCLUSION

Although some of these results already exist for functions of several variables, they have not been proved for real-valued functions of single variable. This could be as a result of the assumption that analysis on $\mathbb{R}^n$ generalizes those on $\mathbb{R}$. As true as this may seem, it is obvious that the results and proofs have relatively deferent make-ups for functions on $\mathbb{R}$. Thus this work provides a very good base for understanding the concept of quasiconcavity.

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