ON THE SOLUTIONS OF S-GTR AND D-GTR EQUATIONS OF BONE TISSUE REGENERATION MODELS

Arunaye, F.I.
Department of Mathematics and Computer science, Delta State University, Abraka
Nigeria
ifarunaye@yahoo.com

ABSTRACT
In Arunaye and Ogunsalu (2010), we obtained mathematical models for osteobstruction in bone regeneration mechanism for skeletal tissue engineering procedures. In this paper we obtain the Lie point symmetries and hence the invariant solutions of these mathematical models for Bone tissue regeneration in 1-D systems.

INTRODUCTION
It is well known that the science of biological engineering involving repairs or replacement of tissues or organs by delivering implanted cell, scaffolds, DNA and proteins at surgery have tremendously sustained many lives in recent time (Butler *et al.,* 2000; Salgado et al., 2004). Bone and graft material resorptions (the destruction, disappearance, or dissolution of bone and biodegradable graft materials by biochemical activity) may results to osteobstruction which inhibits the express bone regeneration processes (Ogunsalu *et al.,* 2010) in bone grafting. It is also well known in the literature the complexity of modeling comprehensive models which will accommodate all embracing dimensionality of the human physiological structure for instance n-D (n-dimensions) structure of a continuum bone segment. However the most practicable structures are in the 1-D, 2-D and 3-D.

Solving differential equations which arose from practical real life situations is not an easy task but with the applications of Lie groups' method; a general integration procedure based on the invariance of differential equations under a continuous group of symmetries (Olver, 1993; Bluman and Kumei, 1989; Stephani, 1989). Of course, we need the solutions of differential equations that make sense to advancing technological medical exploits such as the equations of bone tissue regenerations models. Interested researcher (reader) in the details of the models as well as the Lie groups' invariant technique with which we obtained analytical solutions below is referred to Ogunsalu *et al.* (2010), Arunaye and Ogunsalu (2010), Bhatt and Krishnan (1995), Olver (1993) and Arunaye et al. (2011). This paper is organized as follows. In section 2 we introduced the general model called Fisher's equation for gene transportation in fluid mechanics in 1-D which analytical solutions via groups' invariant method have been obtained in Arunaye and Bhatt 2011. In section 3 we present the Lie symmetries, Lie groups and analytical solutions of the single guided tissue regeneration (S-GTR) and summarily present group invariant solutions of (D-GTR) models in 1-D. Finally section 4 presents discussions and concluding remarks.

FISHER'S EQUATION FOR GENE TRANSPORTATION IN FLUID MECHANICS IN 1-D
This model is given by

$$p_t = k p_{xx} + np(1-p) \qquad (2.1)$$

where $p(x,y)$ is the "frequency of the mutant gene" and n is the "intensity of selection in favour of the mutant gene" and k is a real constant. The Lie symmetries are (Arunaye and Bhatt 2011)

$$\Gamma_1 = \partial_y,\ \Gamma_2 = \partial_x,\ \Gamma_3 = x\partial_y - (\tfrac{y}{2k} - \tfrac{np^2}{2k} yx)p\partial_p,$$

$$\Gamma_4 = \exp[-np^2 x]p\partial_p,$$

$$\Gamma_5 = 2x\partial_x + y\partial_y + 2np(1-p)xp\partial_p \quad , \qquad (2.2)$$

$$\Gamma_6 = 4x^2\partial_x + 4xy\partial_y - (\tfrac{y^2}{k} + 2x - \tfrac{np^2}{k}y^2x - 4np(1-p)x\,\alpha_x = k\alpha_{yy} + np(1-p)$$

,

$$\Gamma_\alpha = [\alpha(y,x) - (2np^2 - np)\int \alpha(y,x)\partial x]\partial_p$$

.

The corresponding Lie groups are (Arunaye and Bhatt 2011):

$$G_1: \quad (y+\lambda, x, p),$$

$$G_2: \quad (y, x+\lambda, p),$$

$$G_3: \quad \left(y+\lambda x, x, p\exp[-\tfrac{\lambda}{2k}(y - np^2yx)]\right),$$

$$G_4: \quad (y, x, p\exp[\lambda e^{-np^2x}]),$$

$$G_5: \quad \left(ye^{\lambda}, xe^{2\lambda}, p\exp[\lambda 2np(1-p)x]\right), \quad (2.3)$$

$$G_6: \quad \left(\frac{y}{1-4\lambda x}, \frac{x}{1-4\lambda x}, p\exp[-\lambda\{\tfrac{y^2}{k} + 2x - \tfrac{np^2}{k}y^2x - 4np(1-p)x^2\}]\right)$$

$$G_\alpha: \quad (y, x, p + \lambda[\alpha(y,x) - (2np^2 - np)\int \alpha(y,x)\partial x])$$

The invariant solutions are (Arunaye and Bhatt, 2011):

,

(2.4)

,

,

$$p^{(6)} = \left\{\beta\left(\frac{y}{1-4\lambda x}, \frac{x}{1-4\lambda x}\right) + np(1-p)\frac{x}{1-4\lambda x}\right\} \exp[\lambda\{\tfrac{y^2}{k} + 2x - \tfrac{np^2}{k}y^2x - 4np(1-p)x^2\}]$$

,

$$p^{(\alpha)} = \beta(y,x) + np(1-p) + \lambda[\alpha(y,x) - (2np^2 - np)\int \alpha(y,x)\partial x]$$

;

where

$$\beta(y,x) = -(\tfrac{1}{k}y^2 + 2x - \tfrac{np^2}{k}y^2x - 4np(1-p)x^2)c_6 + 2np(1-p)xc_5 + \exp[-np^2x]c_4 - (\tfrac{1}{2k}y - \tfrac{np^2}{2k}yx)c_3 - (2np - n)\int \alpha(y,x)\partial x\},$$

and c_6, c_5, c_4, c_3 are arbitrary constants. While $\alpha(y,x)$ is arbitrary solution of

MATHEMATICAL MODELS FOR THE BONE TISSUE REGENERATION in 1-D

In the following we present the analytical solutions of the two models in Ogunsalu et al. using Lie groups invariant technique.

S-GTR model

The model system is given as

$$B_t = \nabla\cdot(-D\nabla B) + \tfrac{1-2\xi}{3}R(1 - B/B_*)B. \quad (3.1)$$

Let us rewrite (3.1) as

$$B_t = -DB_{xx} + \mu B(1-B),$$

$$\mu = R\frac{(1-2\zeta)}{3}, \qquad B_* \equiv 1$$

.

It turns out that the model (3.1) takes the form (2.1). Where the bone concentration threshold (B_*) is taken to be unity, B is concentration of bone cells at graft site, R is rate of bone cell proliferation at graft site and the biodegradable membrane/tissue interaction effect is ξ (real number). Thus we obtain the corresponding Lie symmetries, Lie groups and invariant solutions of (3.1) in 1-D are as follow:

Lie symmetries

$$\Gamma_1 = \partial_x,\ \Gamma_2 = \partial_t,\ \Gamma_3 = t\partial_t - (-\tfrac{x}{2D} + \tfrac{\mu B^2}{2D}xt)B\partial_B,$$

$$\Gamma_4 = \exp[-\mu B^2 t]B\partial_B$$

,

$$\Gamma_5 = 2t\partial_t + x\partial_x + 2\mu B(1-B)tB\partial_B \quad (3.2)$$

$$\Gamma_6 = 4t^2\partial_x + 4tx\partial_x - (-\tfrac{x^2}{D} + 2t + \tfrac{\mu B^2}{D}x^2t - 4\mu B(1-B)t^2)B\partial_B$$

,

$$\Gamma_\alpha = [\alpha(x,t) - (2\mu B^2 - \mu B)\int \alpha(x,t)\partial t]\partial_B$$

.

Corresponding Lie groups

$$G_1: \quad (x+\lambda, t, B), \qquad G_2: \quad (x, t+\lambda, B)$$

,

$$G_3: \quad \left(x+\lambda t, t, B\exp[\tfrac{\lambda}{2D}(x - \mu B^2xt)]\right),$$

$$G_4: \quad (x, t, B\exp[\lambda e^{-\mu B^2 t}]),$$

$$G_5: \quad \left(xe^{\lambda}, te^{2\lambda}, B\exp[\lambda 2\mu B(1-B)t]\right), \quad (3.3)$$

$$G_6: \quad \left(\frac{x}{1-4\lambda t}, \frac{t}{1-4\lambda t}, B\exp[-\lambda\{-\tfrac{x^2}{D} + 2t + \tfrac{\mu B^2}{D}x^2t - 4\mu B(1-B)t^2\}]\right)$$

,

$$G_\alpha : \quad (x,t,B+\lambda[\alpha(x,t)-(2\mu B^2-\mu B)\int \alpha(x,t)\partial t]).$$

Invariant solutions

$$B^{(1)} = \beta(x-\lambda,t)+\mu B(1-B)t$$

$$B^{(2)} = \beta(x,t-\lambda)+\mu B(1-B)(t-\lambda) \quad (3.4)$$

$$B^{(3)} = \{\beta(x-\lambda t,t)+\mu B(1-B)t\}\exp[-\tfrac{\lambda}{2D}(x-\lambda t-\mu B^2(x-\lambda t))]$$

$$B^{(4)} = \{\beta(x,t)+\mu B(1-B)t\}\exp[-\lambda e^{-\mu B^2 t}]$$

$$B^{(5)} = \{\beta\left(xe^{-\lambda},te^{-2\lambda}\right)+\mu B(1-B)te^{-2\lambda}\}\exp[-\lambda 2\mu B(1-B)te^{-2\lambda}]$$

$$B^{(6)} = \left\{\beta\left(\frac{x}{1-4\lambda t},\frac{t}{1-4\lambda t}\right)+\mu B(1-B)\frac{t}{1-4\lambda t}\right\} \exp[\lambda\{-\tfrac{x^2}{D}+2t+\tfrac{\mu B^2}{D}x^2t-4\mu B(1-B)t^2\}]$$

$$B^{(\alpha)} = \beta(x,t)+\mu B(1-B)+\lambda[\alpha(x,t)-(2\mu B^2-\mu B)\int \alpha(x,t)\partial t];$$

where

$$\beta(x,t) = -(-\tfrac{1}{D}x^2+2t+\tfrac{\mu B^2}{D}x^2t-4\mu B(1-B)t^2)c_6 + 2\mu B(1-B)tc_5 + \exp[-\mu B^2 t]c_4 - (-\tfrac{1}{2D}x+\tfrac{\mu B^2}{2D}xt)c_3 - (2\mu B-\mu)\int \alpha(x,t)\partial t.$$

D-GTR model

$$B_t = \nabla\cdot(D_2\nabla B)+R_2(1-B/B_*)B+\varepsilon\lambda_\circ \quad (3.5)$$

Similarly, we have the invariant solutions of (3.5) as

$$B^{(1)} = \beta(x-\lambda,t)+[R_2B(1-B)+\eta_\circ]t$$

$$B^{(2)} = \beta(x,t-\lambda)+[R_2B(1-B)+\eta_\circ](t-\lambda), \quad (3.6)$$

$$B^{(3)} = \{\beta(x-\lambda t,t)+[R_2B(1-B)+\eta_\circ]t\} \exp[\tfrac{\lambda}{2D_2}(x-\lambda t-R_2B^2(x-\lambda t))]$$

$$B^{(4)} = \{\beta(x,t)+[R_2B(1-B)+\eta_\circ]t\}\exp[-\lambda e^{-R_2B^2t}]$$

$$B^{(5)} = \{\beta\left(xe^{-\lambda},te^{-2\lambda}\right)+[R_2B(1-B)+\eta_\circ]te^{-2\lambda}\} \exp[-\lambda 2R_2B(1-B)te^{-2\lambda}]$$

$$B^{(6)} = \left\{\beta\left(\frac{x}{1-4\lambda t},\frac{t}{1-4\lambda t}\right)+[R_2B(1-B)+\eta_\circ]\frac{t}{1-4\lambda t}\right\} \exp[\lambda\{\tfrac{x^2}{D_2}+2t-\tfrac{R_2B^2}{D_2}x^2t - 4R_2B(1-B)t^2\}]$$

$$B^{(\alpha)} = \beta(x,t)+[R_2B(1-B)+\eta_\circ]+\lambda[\alpha(x,t) -(2R_2B^2-R_2B)\int \alpha(x,t)\partial t];$$

where

$$\beta(x,t) = -(\tfrac{1}{D_2}x^2+2t-\tfrac{R_2B^2}{D_2}x^2t-4R_2B(1-B)t^2)c_6 + 2R_2B(1-B)tc_5 + \exp[-R_2B^2t]c_4 - (\tfrac{1}{2D_2}x-\tfrac{R_2B^2}{2D_2}xt)c_3 - (2R_2B-R_2)\int \alpha(x,t)\partial t.$$

DISCUSSION AND CONCLUSION

Guided tissue regeneration (GTR) membrane is a synthetic biodegradable material used to confine bone graft to graft site. The thickness/ type of membrane influences the osteoblastic activities and osseointegration of the graft material at the recipient site (Arunaye and Ogunsalu, 2010). We only considered the models in 1-D (one dimension) for simplicity however; in general the concentration of bone cells at graft site is a function $B(x,t)$, where $x\in\Re^q$, in which case the models are more complicated. In the above, we have used the method of invariant solutions of differential equations to obtain the infinite solutions of both S-GTR and D-GTR models of bone tissue regenerations. The solutions of these models shall enhance the predictability of the state of orthopedic patient in a Hospital.

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