ON THE GENERALIZED CHARACTERIZATION OF THE BETA DISTRIBUTION OF THE FIRST KIND

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ABSTRACT

In this work, we demonstrate that momental measures could be used to characterize the Beta distribution of the first kind. Furthermore, we show that the characterization is unique, and this distribution degenerates into Uniform distribution or the Gama distribution, depending on the choice of its parameters.

Keywords: Beta function, Beta distribution, limiting form, quantiles, points of inflexion

INTRODUCTION

Mathematically, the beta function is defined as

$$B(p,q)=\int_0^1 x^{p-1}(1-x)^{q-1}\,dx \quad \ldots\ldots\ldots \quad \ldots \quad (1)$$

where p and q are positive numbers (Chung, 1979).

By the substitution

$x = sin^2\theta$, $and\ dx = 2\sin\theta\ \cos\theta$, (1)

could also be represented as

$$B(p,q)=2\int_0^{\frac{\pi}{2}} sin^{2p-1}\theta cos^{2q-1}\theta\, d\theta \quad \ldots\ldots\ldots \quad \ldots. \quad (2)$$

Among its basic features we have that B (p, q) = B(q, p) and

$$B(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \quad \ldots \quad \ldots \ldots \ldots \ldots\ldots \ldots\ldots \quad (3)$$

The beta distribution of the first kind is a two-parameter probability distribution based on the definition of the beta function as in (1). Its probability density function is given by

$$f(x)=\frac{1}{B(p,q)}x^{p-1}(1-x)^{q-1}, \quad 0\le x\le 1 \quad \ldots \ldots \ldots (4)$$

Clearly p and q are both positive and the area under the curve $f(x)$ is 1, which means that $f(x)$ is actually a probability density function. It is important to observe that the transformation $y = 1 - x$ produces a mirror image of the original distribution which is still a beta distribution.

Moments and Cumulants of the distribution

An important property of this distribution is that its moments can be easily derived directly, rather than by series expansion. Thus, if x is a random variable with beta distribution, the r^{th} moment of x about zero is

$$\mu_r' = E(x^r) = \frac{1}{B(p,q)}\int_0^1 x^{p+r-1}(1-x)^{q-1}dx \quad \ldots\ldots \quad \ldots\ldots \quad \ldots \quad (5)$$

$$=\frac{B(p+r,q)}{B(p,q)}=\frac{\Gamma(p+r)\Gamma(p+q)}{\Gamma(p)\Gamma(p+r+q)}$$

$$\therefore \mu_1'=\frac{\Gamma(p+1)\Gamma(p+q)}{\Gamma(p)\Gamma(p+q+1)}=\frac{p\Gamma(p).\Gamma(p+q)}{\Gamma(p)(p+q)\Gamma(p+q)}=\frac{p}{p+q}$$

Using the standard factorial notation, we have

$$\mu_1'=\frac{(p+r-1)^{[r]}}{(p+q+r-1)^{[r]}} \quad \ldots\ldots \quad \ldots\ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots\ldots \quad (6)$$

where

$$x^{[r]} = x(x-1)\ldots\ldots\ldots\ldots(x-r+1)$$

hence

$$\mu_2'=\frac{p(p+1)}{(p+q)(p+q+1)}$$

$$\mu_3'=\frac{(p+2)^{[3]}}{(p+q+2)^{[3]}}=\frac{p(p+1)(p+2)}{(p+q)(p+q+1)(p+q+2)}$$

$$\mu_4'=\frac{(p+3)^{[4]}}{(p+q+3)^{[4]}}=\frac{p(p+1)(p+2)(p+3)}{(p+q)(p+q+1)(p+q+2)(p+q+3)}$$

This gives us the first four moments about zero.

Now for the cumulants, we shall apply the following relations (Stuart and Ord, 1998)

$k_1 = \mu_1' = \frac{p}{p+q}$,

$k_2 = \mu_2' - (\mu_2')^2$, $k_3 = \mu_3' - 3\mu_1'\mu_2' + 2(\mu_1')^2$

And $k_4 = \mu_4 - 3\mu_2^2$ where μ_r stands for the r^{th} central moment.

Substituting we get $k_1 = \frac{p}{p+q}$

$$k_2 = \frac{pq}{(p+q)^2(p+q+1)}$$

$$k_3 = \frac{2pq(q-p)}{(p+q)^3(p+q+2)(p+q+1)}$$

And

$$k_4 = \frac{6pq\{(p+q+1)(p-q)^2 - pq(p+q+2)\}}{(p+q)^4(p+q+1)^2(p+q+2)(p+q+3)}$$

Observe that when $p = q = 1$,

$k_1 = \frac{1}{2}$, $k_2 = \frac{1}{12}$, $k_3 = 0$, $k_4 = -\frac{1}{120}$ … and so on.

Shape of Beta distribution

According to Durret (2004), Fisher's coefficients of skewness, Υ_1, and kurtosis, Υ_2, may easily be obtained as follows:-

$$\Upsilon_1 = \frac{k_3}{(k_2)^{3/2}} \quad \cdots \quad \cdots \quad (7)$$

$$= \frac{2pq(q-p)}{(p+q)^3(p+q+1)(p+q+2)} \cdot \sqrt{\frac{(p+q)^6(p+q+1)^3}{p^3q^3}}$$

$$= \frac{2(q-p)}{p+q+2}\left[\frac{p+q+1}{pq}\right]^{1/2}$$

$$\Upsilon_2 = \frac{k_4}{(k_2)^2} \quad \cdots\cdots\cdots \quad \cdots\cdot \quad (8)$$

$$= \frac{6pq[(p+q+1)(p-q)^2 - pq(p+q+2)]}{(p+q)^4(p+q+1)^2(p+q+2)(p+q+3)} \cdot \frac{(p+q)^4(p+q+1)^2}{p^2q^2}$$

$$= \frac{6[(p+q+1)(p-q)^2 - pq(p+q+2)]}{pq(p+q+2)(p+q+3)}$$

When $p = q = 1, \Upsilon_1 = 1$ while $\Upsilon_2 = -6/5$. Here the Beta distribution degenerates into the uniform distribution with $f(x) = 1, 0 \le x \le 1$

When p is finite and $q \to \infty$, then $\Upsilon_1 = \frac{2}{\sqrt{p}}$ while $\Upsilon_2 \to 6/p$, Which are the coefficients for a gamma distribution.

So we may now investigate the nature of the beta distribution when $q \to \infty$.

Now let $z = qx, then\ x = \frac{z}{q}\ and\ dx = \frac{1}{q}dz$

From (1) and (4)

$$f(z) = \frac{1}{B(p,q)}\left(\frac{z}{q}\right)^{p-1}\left(1-\frac{z}{q}\right)^{q-1} \cdot \frac{1}{q}dz$$

$$= \frac{\Gamma(p+q)}{q^p\Gamma(q)} \cdot \frac{1}{\Gamma(p)} z^{p-1}\left(1-\frac{z}{q}\right)^{q-1} dz \quad \ldots \quad (9)$$

Now as $q \to \infty, \left(1-\frac{z}{q}\right)^{q-1} \to e^{-z}$ also by Stirling's second approximation (Stuart and Ord, 1998; Rahman, 1978).

$$\frac{\Gamma(p+q)}{\Gamma(q)} \approx q^p e^{p(p-1)/2q}$$

And as $q \to \infty$

$$, \frac{\Gamma(p+q)}{\Gamma(q)} \approx q^p, \quad \cdots\cdots \quad \cdots\cdots \quad \cdots (10)$$

so that

$$\lim_{q\to\infty} \frac{\Gamma(p+q)}{q^p\Gamma(q)} = 1$$

.

Hence the limiting distribution of z becomes

$$f(z) = \frac{1}{\Gamma(p)} e^{-z} z^{p-1} dz\ for\ 0 \le z \le \infty, \quad \cdots\cdots \quad (11)$$

Which is the gamma distribution. Hence the gamma distribution is the limiting form of the beta distribution as $q \to \infty$ while p is kept finite.

The Quartiles, Mode and Points of Inflexion

From (4) we can derive the three quartiles - quartiles, deciles and percentiles - of the beta distribution. In terms of percentiles, the nth percentile, π_n, of the distribution of *x* is obtained from the equation.

$$\frac{1}{B(p,q)}\int_0^{\pi_n} X^{p-1}(1-x)^{q-1}dx = {}^{n}/_{100} \quad \dots \dots \quad (12)$$

The limits of integration in (12) leads us to the concept of the incomplete Beta function (Athreya and Lahiri, 2006) which is defined as

$$B_y(p,q) = \frac{1}{B(p,q)}\int_0^y x^{p-1}(1-x)^{q-1}dx\,, \quad \dots \quad \dots \quad (13)$$

With $0 < y < 1$. Hence the nth percentile, π_n, is simply obtained as

$$B_{\pi_n}(p,q) = {}^{n}/_{100}.$$

Consequently the common quartiles are $B_{\pi_{25}}(p,q) = 0.25$ for the first quartile $B_{\pi_{50}}(p,q) = 0.5$ for the median and $B_{\pi_{75}}(p,q) = 0.75$ for the 3rd quartile

To obtain the mode of the distribution of x, we have

$$\log f(x) = A + (p-1)\log x + (q-1)\log(1-x) \; (for\ A\ constant)$$

$$\frac{d\log f(x)}{dx} = \frac{p-1}{x} - \frac{q-1}{1-x} \quad \dots \quad \dots \quad (14)$$

$$\frac{d^2\log f(x)}{dx^2} = \frac{p-1}{x^2} - \frac{q-1}{(1-x)^2} \quad \dots \quad \dots \quad (15)$$

At

$$\frac{df}{dx} = 0, \frac{p-1}{x} = \frac{q-1}{1-x} \Rightarrow x = \frac{p-1}{p+q-2} \quad \dots \quad \dots \quad \dots \quad (16)$$

Within the range $0 \le x \le 1,$ therefore either both p and q are greater than 1 or both are less than 1. Put (16) in the second derivative to get

$$\frac{d^2f(x)}{dx^2} = -\frac{(p+q-2)^3}{(p-1)(q-1)}$$

Hence, if both *p* and *q* are less than 1 then $\frac{d^2f}{dx^2} > 0$.

That means the point $x = (p-1)/(p+q-2)$ gives a minimum (anti-mode).

The resulting beta distribution is U-shaped.

If both *p* and *q* are greater than 1, then $\frac{d^2f}{dx^2} < 0$, implying that $x = (p-1)/(p+q-2)$ is the mode of the distribution.

Furthermore if $\frac{d^2f}{dx^2} = 0$, the result is quadratic whose roots are $\frac{p-1}{p+q-2} \pm D,$ where the discriminant D is given by

$$D = \sqrt{\frac{(p-1)(q-1)(p+q-3)}{(p+q-2)^2(p+q-3)^2}}$$

Hence, there are two points of inflexion, symmetrically placed on either sides of the mode at a distance of *D*, provided both parameters are greater than 2. If one of the parameters is greater than 2 and the other equal to 2, then there is only a point of inflexion at either $x = 2/q$ or $x = (p-2)/p$, as $p = 2, q > 2 \ or\ p > 2, q = 2$ respectively.

CONCLUSION

In conclusion, the beta distribution can take various forms for varying values of *p* and *q*. Most importantly, the following are clear.

If $p < 1 \ and\ q < 1$, the distribution is U-shaped, with an anti-mode at $x = (p-1)/(p+q-2)$.

If one of the parameters is less than 1 and the other greater than 1, then the distribution is J-shaped tailing towards 0 or 1 as $q < 1 \ or\ p < 1$ respectively.

If $p = q = 1$, then the distribution in uniform and defined in a unit square of the positive quadrant.

If one of the parameters is equal to 1 and the other equal to 2, the distribution is semi-triangular with area confined to a right angled triangle with base on the *x-axis*.

If $p = 1,\ q = 2 \text{ then } f(x) = 2(1-x) \text{ with } x = 0\ y = 0 \text{ and } y = 2(1-x)$ as boundaries.

If $p = 2, q = 1, \text{then } f(x) = 2x \text{ with } y = 0, x = 1 \text{ and}$

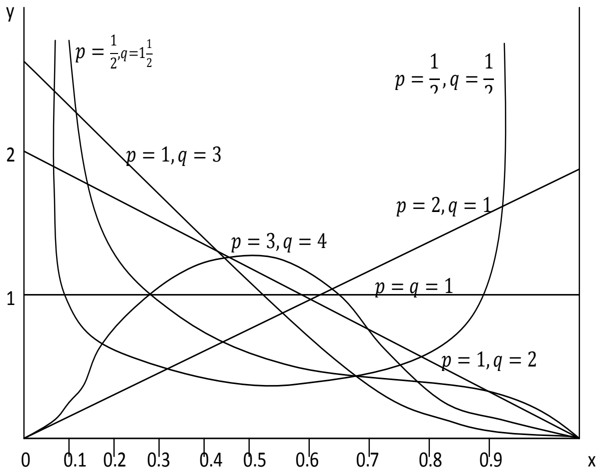

Fig 1. Typical forms of the Beta Distribution of the first kind

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