ON A COUPLED NONLINEAR DIFFERENTIAL EQUATION OF AN INVISCID FLOW: PHYSICAL PROPERTIES

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ABSTRACT
In this paper, the physical properties of a coupled nonlinear differential equation of aninviscid flow at different values of Mach angle α, is presented. The governing partial differential equation is reduced using dimensionless variables into a system of coupled nonlinear differential equation, thereby, establishing the physical properties of existence and uniqueness of solutions and analyzing the stability using Liapunov function to obtain an asymptotic value.

Keywords: Differential Equation, Boundary layer, stability, existence and uniqueness and Liapunov function.

INTRODUCTION
Physical properties of an incompressible boundary layer inviscid flow cannot be over-emphasized in the theory and applications of differential equations due to its numerous applications in the Industrial, Scientific and Engineering sectors. Many authors have done excellent works in identifying the importance of these physical properties to differential equations. Lasota *et al*. (1991), looked into the stability properties of proliferatively coupled cell replication models, while the stability results for the solutions of a certain third order nonlinear differential equation was studied by Ademola *et al.* (2008).Stability and ultimate boundedness of solutions to certain third-order differential equations was studied by Ademola *et al.* (2008). Hua-Shu (2011) studied stability of rotating viscous and inviscid flows and his result showed that inviscid flow is unstable if the velocity profile has an inflection point in parallel flows and stable if it has no inflection point in parallel flow.The linear stability of high-frequency oscillatory flow in a channel was studied by Blennerhassett and Bassom (2006). With respect to our observations in relevant literatures, works on the existence and uniqueness and stability of coupled nonlinear ordinary differential equations are scares. The purpose of this paper is to study the importance of physical properties for solutions of nonlinear differential equations. More so, the existence and uniqueness of solutionswould be established and we would use Liapunov second method as a tool to achieve the desired stability result for the coupled nonlinear differential equation.

MATHEMATICAL MODEL
Let us consider the problem of a laminar boundary layer compressible steady flow [1]:

$$\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)=0 \qquad 2.1$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \qquad 2.2$$

$$\rho C_p\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)-u\frac{\partial p}{\partial x}=\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\mu\left(\frac{\partial u}{\partial y}\right)^2 \qquad 2.3$$

subject to

$$\begin{aligned} &u=v=0, \quad T=T_w \quad at \quad y=0\\ &u=U_1, \quad T=T_1 \quad at \quad y=\infty \end{aligned} \qquad 2.4$$

where the (x,y) are the Cartesian coordinates with $x-$ and $y-$ axes along and normal to the surface of the cylinder respectively, (u,v) are the velocity components along $x-$ and $y-$ axes, ρ is the density, k is the thermal conductivity, C_p is the specific heat at constant pressure. Using the dimensionless variables;

$$\left.\begin{aligned} &\xi = \frac{x}{a}, \quad \eta = Y\left(\frac{\sqrt{v_0}}{R}\right)\mathrm{Re}^{1/2} \\ &\psi = \sqrt{v_0}\,\mathrm{Re}^{1/2}\,\xi f(\xi,\eta), \quad S(x,Y) = S(\xi,\eta) \end{aligned}\right\} \tag{2.5}$$

where ψ is the stream function defined by $u = \partial\psi/\partial y$ and $v = \partial\psi/\partial x$, f and S are the dimensionless functions dependent on η. Using (2.5), equations (2.1) – (2.3), subject to (2.4) is transformed to a system of coupled nonlinear differential equation

$$f''' + ff'' - f'^2 + S + 1 = 0 \qquad 2.6$$

$$kS'' + k'S' + fS' = 0 \qquad 2.7$$

subject to

$$\begin{aligned} &f(0) = f'(0) = 0, \qquad S(0) = S_w \qquad at \quad \eta = 0 \\ &f'(\infty) = 1, \qquad S(\infty) = 0 \qquad when \quad \eta \to \infty \end{aligned} \qquad 2.8$$

where prime denotes differentiation with respect to η and S is the temperature function. Let the thermal conductivity (k) with a very small Mach angle $\alpha << 1$, be

$$k = 1 + \alpha S \qquad 2.9$$

inputting (2.9) into (2.7), we have a new nonlinear differential equation as

$$f''' + ff'' - f'^2 + S + 1 = 0 \qquad 2.10$$

$$(1 + \alpha S)S'' + (\alpha S' + f)S' = 0 \qquad 2.11$$

subject to the condition (2.8).

SOLUTION PROCEDURE

We want to establish the theorem and conditions for existence and uniqueness of solution, and stability of solution by Liapunov second method for the coupled nonlinear differential equations (2.10) and (2.11).

THEOREM I

A function f satisfies a Lipschitz condition on the interval $[a,b]$ such that if f and S be continuous function of ξ and η for all points in some neighbourhood of η_0, then,

$$\|f(\eta) - f(\eta_0)\| \le a\|\eta - \eta_0\| \le b \qquad \forall \quad \eta, \eta_0 \text{ on } [a,b] \qquad 3.1$$

PROOF: Conditions for the existence and uniqueness of solution must be established. Hence, let

$$\frac{\partial f_i}{\partial x_j} \le k \text{ ;such that } i,j = 1(1)6 \qquad 3.2$$

Then there exist a constant $k < \infty$ where

$$k = \max\{k_{ij}\} \qquad 3.3$$

Let

$$x_1 = \eta, x_2 = f, x_3 = f', x_4 = f'', x_5 = S, x_6 = S' \qquad 3.4$$

Therefore,

$$\begin{aligned} x_1' &= 1 \\ x_2' &= f' \\ x_3' &= f'' \\ x_4' &= f''' = -x_2x_4 + (x_3)^2 - x_5 - 1 \\ x_5' &= S' \\ x_6' &= S'' = -\frac{(\alpha x_6 + x_2)x_6}{(1 + \alpha x_5)} \end{aligned} \qquad 3.5$$

and represented by

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \\ x_5' \\ x_6' \end{pmatrix} = \begin{pmatrix} 1 \\ x_3 \\ x_4 \\ -x_2x_4 + (x_3)^2 - x_5 - 1 \\ x_6 \\ -\frac{(\alpha x_6 + x_2)x_6}{(1 + \alpha x_5)} \end{pmatrix} \qquad 3.6$$

satisfying the boundary conditions

$$\begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ x_4(0) \\ x_5(0) \\ x_6(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \beta \\ 0 \\ \omega \end{pmatrix} \qquad 3.7$$

where β and ω are guess until the boundary conditions are satisfied. Equations (3.5) and (3.6) can also be written as;

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \\ x_5' \\ x_6' \end{pmatrix} = \begin{pmatrix} f_1(x_1,x_2,x_3,x_4,x_5,x_6) \\ f_2(x_1,x_2,x_3,x_4,x_5,x_6) \\ f_3(x_1,x_2,x_3,x_4,x_5,x_6) \\ f_4(x_1,x_2,x_3,x_4,x_5,x_6) \\ f_5(x_1,x_2,x_3,x_4,x_5,x_6) \\ f_6(x_1,x_2,x_3,x_4,x_5,x_6) \end{pmatrix} = \begin{pmatrix} 1 \\ x_3 \\ x_4 \\ -x_2x_4 + (x_3)^2 - x_5 - 1 \\ x_6 \\ -\frac{(\alpha x_6 + x_2)x_6}{(1+\alpha x_5)} \end{pmatrix} \qquad 3.8$$

by equations (3.2),

$$\left|\frac{\partial f_i}{\partial x_j}\right| \le k; \quad \forall \quad i,j = 1(1)6$$

$$\left|\frac{\partial f_1}{\partial x_1}\right| = \left|\frac{\partial f_1}{\partial x_2}\right| = \left|\frac{\partial f_1}{\partial x_3}\right| = \left|\frac{\partial f_1}{\partial x_4}\right| = \left|\frac{\partial f_1}{\partial x_5}\right| = \left|\frac{\partial f_1}{\partial x_6}\right| = 0$$

$$\left|\frac{\partial f_2}{\partial x_1}\right| = \left|\frac{\partial f_2}{\partial x_2}\right| = \left|\frac{\partial f_2}{\partial x_4}\right| = \left|\frac{\partial f_2}{\partial x_5}\right| = \left|\frac{\partial f_2}{\partial x_6}\right| = 0; \quad \left|\frac{\partial f_2}{\partial x_3}\right| = 1$$

$$\left|\frac{\partial f_3}{\partial x_1}\right| = \left|\frac{\partial f_3}{\partial x_2}\right| = \left|\frac{\partial f_3}{\partial x_3}\right| = \left|\frac{\partial f_3}{\partial x_5}\right| = \left|\frac{\partial f_3}{\partial x_6}\right| = 0; \quad \left|\frac{\partial f_3}{\partial x_4}\right| = 1$$

$$\left|\frac{\partial f_4}{\partial x_1}\right| = \left|\frac{\partial f_4}{\partial x_6}\right| = 0; \quad \left|\frac{\partial f_4}{\partial x_2}\right| = |-x_4| = n_1;$$

$$\left|\frac{\partial f_4}{\partial x_3}\right| = |2x_3| = n_2; \quad \left|\frac{\partial f_4}{\partial x_4}\right| = |-x_2| = n_3;$$

$$\left|\frac{\partial f_4}{\partial x_5}\right| = |-1| = 1$$

$$\left|\frac{\partial f_5}{\partial x_1}\right| = \left|\frac{\partial f_5}{\partial x_2}\right| = \left|\frac{\partial f_5}{\partial x_3}\right| = \left|\frac{\partial f_5}{\partial x_4}\right| = \left|\frac{\partial f_5}{\partial x_5}\right| = 0; \quad \left|\frac{\partial f_5}{\partial x_6}\right| = 1$$

$$\left|\frac{\partial f_6}{\partial x_1}\right| = \left|\frac{\partial f_6}{\partial x_3}\right| = \left|\frac{\partial f_6}{\partial x_4}\right| = 0; \quad \left|\frac{\partial f_6}{\partial x_2}\right| = \left|\frac{-x_6}{1+\alpha x_5}\right| = n_4;$$

$$\left|\frac{\partial f_6}{\partial x_5}\right| = \left|\frac{\alpha(\alpha x_6 + x_2)x_6}{(1+\alpha x_5)^2}\right| = n_{5;}$$

$$\left|\frac{\partial f_6}{\partial x_6}\right| = \left|\frac{-((\alpha x_6 + x_2) + \alpha x_6)}{1+\alpha x_5}\right| \le \left|\frac{\alpha x_6 + x_2}{1+\alpha x_5}\right| + \left|\frac{\alpha x_6}{1+\alpha x_5}\right| < 2\alpha|x_6| + |x_2| < n_6$$

Clearly, $\left|\frac{\partial f_i}{\partial x_j}\right|_{i,j=1(1)6}$ is bounded and there exists k such that $k = \max\{0,1,n_1,n_2,n_3,n_4,n_5,n_6\}$ and $0 < k < \infty$. Therefore, $f_i(x_1,x_2,x_3,x_4,x_5,x_6)$ are Lipchitz continuous. Hence, there exists a unique solution for the system of equation.

THEOREM 2

Suppose $x = 0$ is a stationary point for $\dot{x} = f(x)$, and the Jacobian matrix A be a Liapunov function such that if

$$\dot{A}(x) \le 0, \quad \forall \quad x \qquad 3.9$$

thenthe eigenvalues λ is asymptotically stable when $x \le 0$ at a very small Mach angle α. Employing Liapunov second theorem, it requires that if all eigenvalues λ of a Jacobian Matrix A have real parts, then, to an arbitrary negative definite quadratic form $(x, W(x))$ with $x = x(t)$, there corresponds a positive definite quadratic form $(x, V(x))$ such that if

$$\frac{dx}{dt} = A(x) \qquad 3.10$$

then, $(x, V(x))$ and $(x, W(x))$, satisfies

$$\frac{d}{dt}(x, V(x)) = (x, W(x)) \qquad 3.11$$

PROOF: Considering the system of differential equation

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = -x_1 x_3 + x_2^2 - y_1 - 1$$
$$\dot{y}_1 = y_2$$
$$\dot{y}_2 = -(1+\alpha y_1)^{-1}(\alpha y_2 + x_1)y_2 \qquad 3.12$$

with stationary points at $\dot{x}_3 = 0 \; and \; \dot{y}_2 = 0$, we assume four stationary points, say; $(x_1, x_2, x_3, y_1, y_2) = (0,0,0,0,0), (1,0,1,0,2), (1,-2,1,1,-2)$ and $(1,1,1,1,1)$, such that for a very small Mach angle $\alpha = 0.3$, $\forall \; \lambda \le 0$, we have the Jacobian Matrix thus;

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -x_3 & 2x_2 & -x_1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -\frac{y_2}{(1+\alpha y_1)} & 0 & 0 & \frac{\alpha y_2(\alpha y_2 + x_1)}{(1+\alpha y_1)^2} & \frac{-(2\alpha y_2 + x_1)}{(1+\alpha y_1)} \end{pmatrix} \qquad 3.13$$

Hence, the eigenvalues of the matrix at the four stationary points would be generated depending on the value of α. Hence, rewriting (3.13),we have

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -x_3 & 2x_2 & -x_1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -\frac{y_2}{(1+0.3y_1)} & 0 & 0 & \frac{0.09y_2^2 + 0.3x_1 y_2}{(1+0.3y_1)^2} & -\frac{(0.6y_2 + x_1)}{(1+0.3y_1)} \end{pmatrix} \qquad 3.14$$

at $(0,0,0,0,0)$, the matrix (3.14) becomes

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

such that $(A - \lambda I)$ gives

$$(A - \lambda I) = \begin{pmatrix} -\lambda & 1 & 0 & 0 & 0 \\ 0 & -\lambda & 1 & 0 & 0 \\ 0 & 0 & -\lambda & -1 & 0 \\ 0 & 0 & 0 & -\lambda & 1 \\ 0 & 0 & 0 & 0 & -\lambda \end{pmatrix} = 0$$

$$-\lambda\left|-\lambda\left|-\lambda\begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix}\right|\right| \pm 0 = 0$$

$$-\lambda^5 = 0$$

$$\lambda = 0$$

(positive definite)

For points $(1,0,1,0,2)$, the matrix (3.14) is

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -2 & 0 & 0 & \frac{24}{25} & -\frac{11}{5} \end{pmatrix}$$

$$(A - \lambda I) = \begin{pmatrix} -\lambda & 1 & 0 & 0 & 0 \\ 0 & -\lambda & 1 & 0 & 0 \\ -1 & 0 & -1-\lambda & -1 & 0 \\ 0 & 0 & 0 & -\lambda & 1 \\ -2 & 0 & 0 & \frac{24}{25} & -\frac{11}{5}-\lambda \end{pmatrix} = 0$$

$$-\lambda\left|-\lambda\left|(-1-\lambda)\begin{vmatrix} -\lambda & 1 \\ \frac{24}{25} & -\frac{11}{5}-\lambda \end{vmatrix}\right|\right| \pm 0 = 0$$

$$-\lambda^5 - \frac{16}{5}\lambda^4 - \frac{31}{25}\lambda^3 + \frac{24}{25}\lambda^2 = 0$$

$$(\lambda + 1)(25\lambda^4 - 55\lambda^3 + 24\lambda^2) = 0$$

$$\Rightarrow \quad \lambda = -1$$

(negative definite)

At stationary points $(1,-2,1,1,-2)$, the matrix (3.14) is

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & -4 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \frac{20}{13} & 0 & 0 & -\frac{24}{169} & \frac{2}{13} \end{pmatrix}$$

$$(A-\lambda I) = \begin{pmatrix} -\lambda & 1 & 0 & 0 & 0 \\ 0 & -\lambda & 1 & 0 & 0 \\ -1 & -4 & -1-\lambda & -1 & 0 \\ 0 & 0 & 0 & -\lambda & 1 \\ \frac{20}{13} & 0 & 0 & -\frac{24}{169} & \frac{2}{13}-\lambda \end{pmatrix} = 0$$

$$-\lambda\left|-\lambda\left|(-1-\lambda)\begin{vmatrix} -\lambda & 1 \\ -\frac{24}{169} & \frac{2}{13}-\lambda \end{vmatrix}\right|\right| \pm 0 = 0$$

$$-\lambda^5 - \frac{11}{13}\lambda^4 + \frac{50}{169}\lambda^3 + \frac{24}{169}\lambda^2 = 0$$

$$(\lambda+1)(169\lambda^4 + 26\lambda^3 + 24\lambda^2) = 0$$

$$\Rightarrow \quad \lambda = -1$$

(negative definite)

for points $(1,1,1,1,1)$, we have the matrix (3.14) as

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -\frac{10}{13} & 0 & 0 & \frac{3}{13} & -\frac{16}{13} \end{pmatrix}$$

$$(A-\lambda I) = \begin{pmatrix} -\lambda & 1 & 0 & 0 & 0 \\ 0 & -\lambda & 1 & 0 & 0 \\ -1 & 2 & -1-\lambda & -1 & 0 \\ 0 & 0 & 0 & -\lambda & 1 \\ -\frac{10}{13} & 0 & 0 & \frac{3}{13} & -\frac{16}{13}-\lambda \end{pmatrix} = 0$$

$$-\lambda\left|-\lambda\left|(-1-\lambda)\begin{vmatrix} -\lambda & 1 \\ \frac{3}{13} & -\frac{16}{13}-\lambda \end{vmatrix}\right|\right| \pm 0 = 0$$

$$-\lambda^5 - \frac{29}{13}\lambda^4 + \frac{23}{13}\lambda^3 + \lambda^2 = 0$$

$$-13\lambda^5 - 29\lambda^4 + 23\lambda^3 + 13\lambda^2 = 0$$

$\lambda = 0$ (positive definite)

our results shows that for every corresponding positive definite eigenvalues, there is a corresponding negative definite eigenvalues, indicating that our Liapunov function is asymptotically stable for all values of $\lambda = 0 \text{ or } -1$.

CONCLUSION

We have been able to establish the conditions for the existence and uniqueness of solution for equations (2.10) and (2.11) subject to (2.8) and it was deduced to be Lipchitz continuous.Also, employing Liapunov function to the coupled nonlinear differential equation of theinviscid flow, our result showed that the function is asymptotically stable and our results improved on some well-known result in the literature.

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